الجمهورية الجزائرية الديمقراطية الشعبية

People's Democratic Republic of Algeria

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

University Center Abdelhafid Boussouf – Mila

المركز الجامعي
عبد الحفيظ بوالصوف ميلة

Institute: Mathematics and Computer Sciences

Department: Mathematics

Order N° : …………….

Matricule : M103/2021

www.centre-univ-mila.dz

Thesis

Submitted in partial fulfilment of the requirements
for the degree of Doctorate 3^{th} cycle (LMD)

Dynamic behavior of multidimensional systems of difference equations

Field: Mathematics

Specialty: Dynamical systems

Presented and defended by: Allam Asma

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University year: 2023/2024

Acknowledgements

This thesis was only made possible with the assistance of several individuals, to whom I wish to extend my heartfelt appreciation. I am deeply thankful to all the individuals who played a crucial role in the completion of this thesis. Each of them contributed significantly and was instrumental in the success of this academic project.

First and foremost, I would like to extend my heartfelt thanks to my thesis advisor, Mr.Yacine Halim. His constant availability, and insightful guidance have been invaluable throughout my research journey. Thanks to his wise counsel, I was able to overcome obstacles and make significant progress in my reflections. His contribution to this thesis goes far beyond his role as an academic supervisor, and I am profoundly grateful to him.

I also want to express my gratitude to the members of the jury for their interest in my research work. Their relevant comments and constructive suggestions helped enrich my thinking and improve the quality of my thesis. Their commitment to academic excellence has been a source of inspiration throughout this process.

A big thank to the administrators of the doctoral program at the University Center of Mila for their continuous support. Their expertise and dedication have helped create an environment conducive to research and learning, where I could develop my skills and broaden my academic horizons.

I also wish to convey my gratitude to my fellow doctoral students, whose collaboration and exchange of ideas enriched my academic experience. Their mutual support and camaraderie were a source of comfort and motivation throughout this demanding journey.

Finally, my sincere thanks go to all the individuals who contributed directly or indirectly to the completion of this thesis. Their support, encouragement, and commitment were crucial to achieving this outcome. I am deeply grateful to them for their valuable assistance and trust in me.

Abstract

In the realm of dynamical systems, examining how solutions of systems of difference equations behave over time holds profound significance as it unveils the underlying patterns and trajectories that guide the evolution of various systems. This thesis delves into finding the form of solutions for specific multidimensional systems of difference equations and studying their behavior. Specifically, we are interested in a discrete community model, the form and the asymptotic behavior of solutions to a close-to-cyclic multidimensional difference equations system, and the convergence of solutions of a two-dimensional system of higher-order difference equations.

Several results are then presented about the form of solutions, the asymptotic behavior, the global attractivity, the rate of convergence, and the convergence of solutions, in addition to numerous simulations which allow confirming and bringing out our contributions.

Keywords: Systems of difference equations, equilibrium points, qualitative study, local stability, asymptotic behavior, rate of convergence.

ملخص

في مجال الأنظمة الديناميكية، تحمل دراسة السلوك المقارب لحلول معادلات الفروق أهمية عميقة لأنها تكشف النقاب عن الأنماط والمسارات الأساسية التي توجه تطور الأنظمة المختلفة. تعنى هذه الأطروحة بإيجاد ودراسة سلوك الحلول لبعض الأنظمة متعددة الأبعاد لمعادلات الفروق. على وجه التحديد، نحن مهتمون بنموذج مجتمعي منفصل وبشكل الحلول والسلوك المقارب لنظام متعدد الأبعاد قريب من الدوري لمعادلات الفروق، وتقارب حلول نظام ثنائي الأبعاد من معادلات الفروق ذات الرتب العليا.

يتم تقديم العديد من النتائج حول شكل الحلول، السلوك المقارب، الجذب العام، تقارب الحلول، ومعدل التقارب، بالإضافة إلى العديد من عمليات المحاكاة التي تسمح بتأكيد وإبراز مساهماتنا.

الكلمات الأساسية: أنظمة معادلات الفروق، نقاط التوازن، الدراسة النوعية، الاستقرار المحلي، السلوك المقارب، معدل التقارب.

Résumé

Dans le domaine des systèmes dynamiques, l'étude du comportement asymptotique des solutions des systèmes d'équations aux différences revêt une signification profonde, car elle révèle les schémas sous-jacents et les trajectoires qui guident l'évolution de divers systèmes. Cette thèse se consacre à la recherche et à l'étude du comportement des solutions de certains systèmes multidimensionnels d'équations aux différences. Plus précisément, nous nous intéressons à un modèle discret de communauté, à la forme et au comportement asymptotique des solutions d'un système multidimensionnel proche du cyclique d'équations aux différences, ainsi qu'à la convergence des solutions d'un système bidimensionnel d'équations aux différences d'ordre supérieur.

Ensuite, plusieurs résultats sont présentés, notamment concernant la forme des solutions, le comportement asymptotique, l'attractivité globale, la convergence des solutions et le taux de convergence. De plus, de nombreuses simulations sont réalisées pour confirmer et mettre en évidence nos contributions.

Mots-clés: Systèmes d'équations aux différences, points d'équilibre, étude qualitative, stabilité locale, comportement asymptotique, ordre de convergence.

Table of contents

General introduction

Researchers and scientists from various fields are becoming increasingly interested in the difference equations theory. Consequently, numerous papers have been published addressing difference equations and systems thereof. Some of them can be found in previous research (see for example [1, 12, 21, 28, 30, 31, 33, 34, 41, 43, 55, 58]).

The primary contribution of this thesis lies in its proposal of closed-form solutions for multidimensional systems of difference equations. This extension to higher dimensions is noteworthy, as many existing methods are tailored for lower-dimensional systems. We effectively demonstrate the novelty of our approach by comparing it to existing literature. The results presented in the thesis are not only new but also capable of generalizing previous findings, thus advancing the current state of research in this area.

In the introductory portion of our first chapter, we lay out fundamental definitions and significant findings relevant to difference equations and the systems they encompass. These foundational concepts serve as a groundwork for us to delve into the core focus of this thesis. Our primary goal involves conducting a thorough qualitative investigation aimed at discovering explicit solutions for specific types of nonlinear difference equations systems. These systems may include multidimensional systems and symmetric systems, which add complexity and richness to our analytical exploration.

Multidimensional difference equations systems are like building blocks for understanding how things change over time. They are used in many areas, like economics to understand money growth, and in biology to study how animals in an environment interact.

In the second section of our first chapter, we bring forth a fresh category of nonlinear difference equations systems characterized by a multitude of interconnected equations arranged in a distinctive manner. We meticulously analyze the structure of the following system, paying close attention to its intricate connections

$$y_{n+1}^{(i)} = \frac{a_i y_n^{(i+1)} \left(y_{n-k}^{(i+1)}\right)^{p_{i+1}} + b_i}{\left(y_{n-k+1}^{(i)}\right)^{p_i}}; \quad n \in \mathbb{N}_0,$$

where $y_n^{(i+k)} = y_n^{(i)}, p_{i+k} = p_i, a_{i+k} = a_i, b_{i+k} = b_i; i = \overline{1,k}$, the initial values $y_{-k}^{(i)}, y_{-k+1}^{(i)}, \ldots, y_0^{(i)}$ and the parameters a_i and b_i, $i = \overline{1,k}$ are positive real numbers and p_i, $i = \overline{1,k}$, are real numbers. Our main emphasis is on unraveling the methods for representing solutions to this complex array of equations. Through detailed examination and scrutiny, we aim to provide insights into the behavior and properties of solutions within this specific framework.

In the study of how things change over time, there is also an important group of systems called symmetric systems of nonlinear difference equations. These systems exhibit a regular pattern and help us to understand how connected things change together. They are used in many domains, ranging from modeling chemical interactions to analyzing the stability of physical systems. For instance, these systems are employed to study synchronized oscillations in neural networks, as well as to analyze collective animal movements in behavioral biology.

Taking inspiration from earlier studies, the second chapter of this thesis delves into a new type of symmetric system involving nonlinear difference equations. This system is characterized by its distinctive symmetry properties, which play a crucial role in its dynamics and behavior and affect how the system's equations work. Through comprehensive analysis and investigation, we want to study the system below carefully to understand it better and show what makes it different

$$x_{n+1}=\frac{x_{n-(2k+1)}}{1+y_{n-k}},\quad y_{n+1}=\frac{y_{n-(2k+1)}}{1+x_{n-k}},\quad n,k\in\mathbb{N}_0,$$

the initial values $x_{-(2k+1)}, x_{-2k}, \ldots, x_0, y_{-(2k+1)}, y_{-2k}, \ldots, y_0$ are non-negative real numbers.

In the world of biology, using difference equations helps us understand how animal populations change. These equations create models that show how animal numbers go up and down over time, and how different species interact. For example, they help researchers study how animals grow, compete for food, and interact with predators.

Additionally, these models help us understand how environmental factors like habitat and reproduction affect animal populations over time. By testing different scenarios and predicting population trends, scientists can plan ways to protect habitats and manage biodiversity.

Studying these models not only helps us understand basic ecological processes but also guides conservation efforts and predicts the effects of environmental changes. In the final chapter, we are going to take a closer look at the following specific type of complex equations system to better understand and manage animal populations in dynamic environments

$$x_{n+1}=\frac{a_1x_n-a_2x_ny_n}{1+a_3x_n},\quad y_{n+1}=\frac{a_4y_n+a_5y_nz_n}{1+a_6y_n},\quad z_{n+1}=\frac{a_7z_n+a_8z_nx_n}{1+a_9z_n},\qquad n\in\mathbb{N}_0,$$

where the parameters a_i, $i=\overline{1,9}$ and the initial values x_0, y_0 and z_0 are positive real numbers.

Chapter 1

Preliminaries and solvability of a multidimensional close-to-cyclic system of difference equations

1.1 Preliminaries

The first part of our opening chapter aims to explain difference equations and their systems in simple terms. We want to make it easy for readers to understand these ideas. Also, we talk about stability, which means whether these equations and systems stay the same or change over time. This helps us see how these math models work in different situations.

Additionally, we talk about specific theorems. These are important ideas that we are going to use a lot in our thesis. They help us analyze and understand the math parts of our research. They give us important rules and ideas to follow in our study.

In summary, this first part of our opening chapter sets the stage for our thesis. We explain key concepts clearly, talk about stability in difference equations and their systems, and introduce important theorems that will help us throughout our research.

For these preliminary elements, we refer to the following references [10], [14],[18], [22] and [41].

This first section of our first chapter gives some definitions and general results concerning equations and systems of difference equations, stability, and theorems that we are going to use in the rest of our thesis.

1.1.1 Linear difference equations

The linear difference equations' study is highly important in applied mathematics. These equations are very useful tools for representing and understanding how things change and vary in different domains. In this part, we are going to explain what they are and present definitions and theorems that will help us better understand the concepts and the methods we will see later on.

Definition 1.1.1 *An equation expressed as*

$$x_{n+k} + p_1(n)\, x_{n+k-1} + \cdots + p_k(n)\, x_n = g(n)\,,\ n \in \mathbb{N}_{n_0} \tag{1.1}$$

is called **Linear difference equation** *of order k as long as* $p_k(n) \neq 0$, *where*

$p_1(n), p_2(n), \ldots, p_k(n), g(n)$ *are well-defined functions on* $\mathbb{N}_{n_0}$.

Remarks 1.1.1
In general, we associate k initial values with equation (1.1).

$$x_{n_0} = c_1, x_{n_0+1} = c_2, \ldots, x_{n_0+k-1} = c_k, \tag{1.2}$$

$c_i,\ i = \overline{1,k}$ *represent real or complex constants.*

Definition 1.1.2 *Equation (1.1) with* $g(n) = 0, \forall n \geq n_0$, *is called* **homogeneous linear**

difference equation, *and it is written as follows*

$$x_{n+k} + p_1(n)\, x_{n+k-1} + \cdots + p_k(n)\, x_n = 0. \tag{1.3}$$

Definition 1.1.3 *A sequence $\{x_n\}_{n \geq n_0}$ is considered a solution to equation (1.1) with the initial values (1.2), if it satisfies relation (1.1) and the initial values (1.2).*

Theorem 1.1.1 [14]

Equation (1.1) with the initial values (1.2) has one and only one solution.

Theorem 1.1.2 [14]

The set S of the solutions to the difference equation (1.3) is a vector space on $\mathbb{K}$ of dimension K.

Definition 1.1.4 *A set of k linearly independent solutions of the difference equation (1.3) is referred to as* **a fundamental set** *of solutions.*

The next theorem illustrates that the homogeneous linear difference equation (1.3) always admits a fundamental set of solutions (i.e. a basis of solutions).

Theorem 1.1.3 [14, 41]

- *If $p_k(n) \neq 0$, for all $n \geq n_0$, the homogeneous linear difference equation (1.3) possesses a fundamental set of solutions.*
- *If $x_n^1, x_n^2, \ldots, x_n^k$ are solutions of equation (1.3), so*

$$x_n = a_1 x_n^1 + a_2 x_n^2 + \cdots + a_k x_n^k$$

is also a solution of equation (1.3), where a_i, $i = \overline{1,k}$, are arbitrary constants.

Corollary 1.1.1 *Suppose* $\left\{\left(x_n^1\right)_{n\geq n_0}, (x_n^2)_{n\geq n_0}, \ldots, \left(x_n^k\right)_{n\geq n_0}\right\}$ *form a fundamental set of solutions to equation* (1.3)*. So, the general solution of* (1.3) *is represented*

$$x_n = \sum_{i=1}^{k} a_i x_n^i,$$

where $a_i, i = \overline{1,k}$*, are arbitrary constants.*

Theorem 1.1.4 [14, 41]

Let $\left\{\left(x_n^1\right)_{n\geq n_0}, (x_n^2)_{n\geq n_0}, \ldots, \left(x_n^k\right)_{n\geq n_0}\right\}$ *be a fundamental set of solutions to equation* (1.3) *and* $\left(x_n^p\right)_{n\geq n_0}$ *a particular solution to equation* (1.1)*, then any general solution of equation* (1.1) *takes the form*

$$x_n = \sum_{i=1}^{k} a_i x_n^i + x_n^p, \quad n \geq n_0.$$

Linear difference equations with constant coefficients

In what follows, we focus on homogeneous linear difference equations with constant coefficients, i.e.

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \tag{1.4}$$

$p_i,\ i = \overline{1,k}$ represent real or complex constants.

Resolution of the homogeneous linear difference equations with constant coefficients

Our aim is to identify a fundamental set of solutions and thereby determine the general solution to equation (1.4).

Theorem 1.1.5 [14, 22]

Equation (1.4) *has solutions of the form*

$$x_n = \lambda^n,$$

where $\lambda \in \mathbb{C}^*$, *and it verifies*

$$p(\lambda) = \sum_{i=0}^{k} p_i \lambda^{k-i} = 0. \tag{1.5}$$

with $p_0 = 1$.

Definition 1.1.5 *The polynomial*

$$p(\lambda) = \sum_{i=0}^{k} p_i \lambda^{k-i}$$

with $p_0 = 1$, *is termed* **the characteristic polynomial** *associated with equation* (1.4).

Theorem 1.1.6 [14, 22]

If the roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ *of the characteristic polynomial* $p(\lambda)$ *are distinct, then* $\left\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n\right\}$ *forms a fundamental set of solutions to equation* (1.4).

Corollary 1.1.2 *Any solution of equation* (1.4) *can be expressed as a linear combination of* λ_i^n, *where* $i = \overline{1,k}$, *i.e.*

$$x_n = \sum_{i=1}^{k} c_i \lambda_i^n, c_i \in \mathbb{K},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are distinct roots of* $p(\lambda)$.

Theorem 1.1.7 [14, 22]

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r, r \leq k$, *are the roots of the characteristic polynomial associated to equation* (1.4), *with degrees of multiplicity* $m_1, m_2, \ldots, m_r$ *respectively* $\left(\sum_{i=1}^{r} m_i = k\right)$, *so*

$$\left\{\left(\lambda_1^n\right)_{n\geq n_0}, \left(n\lambda_1^n\right)_{n\geq n_0}, \left(n^2\lambda_1^n\right)_{n\geq n_0}, \ldots, \left(n^{m_1-1}\lambda_1^n\right)_{n\geq n_0}, \left(\lambda_2^n\right)_{n\geq n_0}, \left(n\lambda_2^n\right)_{n\geq n_0}, \left(n^2\lambda_2^n\right)_{n\geq n_0}, \ldots,\right.$$

$$\left.\left(n^{m_2-1}\lambda_2^n\right)_{n\geq n_0}, \ldots, (\lambda_r^n)_{n\geq n_0}, (n\lambda_r^n)_{n\geq n_0}, \left(n^2\lambda_r^n\right)_{n\geq n_0}, \ldots, \left(n^{m_r-1}\lambda_r^n\right)_{n\geq n_0}\right\},$$

is a fundamental set of equation (1.4).

Corollary 1.1.3 [14]

The solution of equation (1.4) *is expressed as*

$$y_n = \sum_{i=1}^{r} \sum_{j=0}^{m_i-1} c_{ij} n^j \lambda_i^n, \;\; c_{ij} \in \mathbb{K},$$

where

- *The parameter $r \leq k$ denotes the number of distinct roots of the characteristic equation* (1.5).
- *The parameter λ_i denotes one of the roots of the characteristic equation* (1.5).
- *The parameter m_i denotes the degree of multiplicity of the root λ_i.*
- *The coefficients c_{ij} are constants determined from the initial values.*

1.1.2 Nonlinear difference equations

Nonlinear difference equations are very useful tools for representing various phenomena in many fields. Unlike linear equations, these have terms that are not linear, which can make them more challenging to understand. In this part, we are going to examine these equations closely.

Assume I is a part of $\mathbb{R}$, and $f : I^{k+1} \longrightarrow I$ is a continuously differentiable function.

Definition 1.1.6 *A difference equation of order* $(k+1)$,

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.6}$$

with $x_0, x_{-1}, \ldots, x_{-k} \in I$, is said to be nonlinear if it is not of the form (1.1).

Definition 1.1.7 *A point $\bar{x} \in I$ is said to be* **an equilibrium point** *of equation* (1.6) *if*

$$\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x}),$$

in other words

$$x_n = \bar{x}, \ \ \forall n \geq -k.$$

Definition 1.1.8 *An interval $J \subseteq I$ is said to be* **an invariant interval** *of equation* (1.6) *if*

$$x_{-k}, x_{-k+1}, \cdots, x_0 \in J \Rightarrow x_n \in J, \ \ n > 0.$$

1.1.3 About stability

If we are unable to find a solution, we resort to a qualitative study, as the most important characteristic that can be studied is stability.

Definition 1.1.9 *Suppose that $\bar{x}$ is an equilibrium point of* (1.6)*,*

1. *$\bar{x}$ is considered* **locally stable** *if*

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall \ x_{-k}, x_{-k+1}, \ldots, x_0 \in I: \ \ |x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta,$$

then

$$|x_n - \bar{x}| < \varepsilon, \ \ \forall n \geq -k.$$

2. *$\bar{x}$ is considered* **locally asymptotically stable** *if*

- *$\bar{x}$ is locally stable.*
- $\exists \gamma > 0, \ \forall \ x_{-k}, x_{-k+1}, \ldots, x_0 \in I: \ \ |x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma$, *so*

$$\lim_{n \to +\infty} x_n = \bar{x}.$$

3. *$\bar{x}$ is considered* **globally attractive** *if*

$$\forall \ x_{-k}, x_{-k+1}, \ldots, x_0 \in I, \ \ \lim_{n \to +\infty} x_n = \bar{x}.$$

4. *$\bar{x}$ is considered* **globally asymptotically stable** *if*

   - *$\bar{x}$ is locally stable.*
   - *$\bar{x}$ is globally attractive.*

5. *$\bar{x}$ is considered* **unstable** *if it lacks local stability.*

Definition 1.1.10 *We call* **linear difference equation associated** *with equation (1.6), the equation of the form below*

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \cdots + p_k y_{n-k}, \tag{1.7}$$

where

$$p_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \ldots, \bar{x}), \ \ for \ \ i = \overline{0, k},$$

and

$$\begin{array}{rccc} f: & I^{k+1} & \longrightarrow & I \\ & (u_0, u_1, \ldots, u_k) & \longmapsto & f(u_0, u_1, \ldots, u_k). \end{array}$$

Theorem 1.1.8 [41] **(Stability by linearization)**

1. *If all the roots of the characteristic polynomial of the associated linear difference equation lie within the open unit disk $|\lambda| < 1$, then the equilibrium point of (1.6) is locally asymptotically stable.*

2. *If there exists at least one root of the characteristic polynomial of the associated linear difference equation with a modulus exceeding one, then the equilibrium point of (1.6) is unstable.*

1.1.4 System of nonlinear difference equations

Suppose $f^{(1)}, f^{(2)}, \ldots, f^{(p)}$ denote functions that are continuously differentiable, such that

$$f^{(i)} : I_1^{k+1} \times I_2^{k+1} \times \cdots \times I_p^{k+1} \to I_i^{k+1}, \ \ i = \overline{1, p},$$

with $I_i,\ i=\overline{1,p}$ present real intervals.

Consider the following p-dimensional system

$$\begin{cases} x_{n+1}^{(1)} & = & f^{(1)}\left(x_n^{(1)},x_{n-1}^{(1)},\ldots,x_{n-k}^{(1)},x_n^{(2)},x_{n-1}^{(2)},\ldots,x_{n-k}^{(2)},\ldots\ldots,x_n^{(p)},x_{n-1}^{(p)},\ldots,x_{n-k}^{(p)}\right) \\ x_{n+1}^{(2)} & = & f^{(2)}\left(x_n^{(1)},x_{n-1}^{(1)},\ldots,x_{n-k}^{(1)},x_n^{(2)},x_{n-1}^{(2)},\ldots,x_{n-k}^{(2)},\ldots\ldots,x_n^{(p)},x_{n-1}^{(p)},\ldots,x_{n-k}^{(p)}\right) \\ & \vdots & \\ x_{n+1}^{(p)} & = & f^{(p)}\left(x_n^{(1)},x_{n-1}^{(1)},\ldots,x_{n-k}^{(1)},x_n^{(2)},x_{n-1}^{(2)},\ldots,x_{n-k}^{(2)},\ldots\ldots,x_n^{(p)},x_{n-1}^{(p)},\ldots,x_{n-k}^{(p)}\right) \end{cases} \tag{1.8}$$

with $n,\ k\in\mathbb{N}_0,\ \left(x_{-k}^{(i)},x_{-k+1}^{(i)},\ldots,x_0^{(i)}\right)\in I_i^{k+1},\ \ i=\overline{1,p}.$

Let's establish the function

$$F: I_1^{(k+1)}\times I_2^{(k+1)}\times\cdots\times I_p^{(k+1)}\longrightarrow I_1^{(k+1)}\times I_2^{(k+1)}\times\cdots\times I_p^{(k+1)}$$

as follow

$$F(X)=\left(f_0^{(1)}(X),f_1^{(1)}(X),\ldots,f_k^{(1)}(X),f_0^{(2)}(X),f_1^{(2)}(X),\ldots,f_k^{(2)}(X),\ldots,f_0^{(p)}(X),f_1^{(p)}(X),\ldots,f_k^{(p)}(X)\right),$$

with

$$X=\left(u_0^{(1)},u_1^{(1)},\ldots,u_k^{(1)},u_0^{(2)},u_1^{(2)},\ldots,u_k^{(2)},\ldots,u_0^{(p)},u_1^{(p)},\ldots,u_k^{(p)}\right)^T,$$

$$f_0^{(i)}(X)=f^{(i)}(X),\quad f_1^{(i)}(X)=u_0^{(i)},\ldots,f_k^{(i)}(X)=u_{k-1}^{(i)},\ \ i=\overline{1,p}.$$

Let's put

$$X_n=\left(x_n^{(1)},x_{n-1}^{(1)},\ldots,x_{n-k}^{(1)},x_n^{(2)},x_{n-1}^{(2)},\ldots,x_{n-k}^{(2)},\ldots,x_n^{(p)},x_{n-1}^{(p)},\ldots,x_{n-k}^{(p)}\right)^T.$$

Thus, system (1.8) can be expressed as the following one

$$X_{n+1}=F(X_n),\ \ n=0,1,2,\ldots \tag{1.9}$$

that's to say

$$\begin{cases} x_{n+1}^{(1)} & = & f^{(1)}\left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots\ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right) \\ x_n^{(1)} & = & x_n^{(1)} \\ & \vdots & \\ x_{n-k+1}^{(1)} & = & x_{n-k+1}^{(1)} \\ x_{n+1}^{(2)} & = & f^{(2)}\left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots\ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right) \\ x_n^{(2)} & = & x_n^{(2)} \\ & \vdots & \\ x_{n-k+1}^{(2)} & = & x_{n-k+1}^{(2)} \\ & \vdots & \\ x_{n+1}^{(p)} & = & f^{(p)}\left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots\ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right) \\ x_n^{(p)} & = & x_n^{(p)} \\ & \vdots & \\ x_{n-k+1}^{(p)} & = & x_{n-k+1}^{(p)} \end{cases}$$

Definition 1.1.11

1. $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$ *is considered* **an equilibrium point** *of system* (1.8) *if*

$$\overline{x^{(1)}} = f^{(1)}\left(\overline{x^{(1)}}, \overline{x^{(1)}}, \ldots, \overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}, \overline{x^{(p)}}, \ldots, \overline{x^{(p)}}\right),$$

$$\overline{x^{(2)}} = f^{(2)}\left(\overline{x^{(1)}}, \overline{x^{(1)}}, \ldots, \overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}, \overline{x^{(p)}}, \ldots, \overline{x^{(p)}}\right),$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\overline{x^{(p)}} = f^{(p)}\left(\overline{x^{(1)}}, \overline{x^{(1)}}, \ldots, \overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}, \overline{x^{(p)}}, \ldots, \overline{x^{(p)}}\right).$$

2. $\overline{X} = \left(\overline{x^{(1)}}, \overline{x^{(1)}}, \ldots, \overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}, \overline{x^{(p)}}, \ldots, \overline{x^{(p)}}\right) \in I_1^{k+1} \times I_2^{k+1} \times \ldots \times I_p^{k+1}$ *represents an equilibrium of system* (1.9) *if*

$$\overline{X} = F(\overline{X}).$$

1.1.5 About stability

The stability of difference equations is a very important aspect when studying them. It concerns how the solutions of a system evolve over time in response to changes or disturbances in the initial conditions or parameters. Studying stability allows us to determine whether a system tends to remain stable, oscillate, or become unstable over time. This helps us evaluate how robust and predictable the system is. By analyzing stability, we can better understand how the system will behave in the long run and predict how it will respond to changes.

Definition 1.1.12 *Suppose $\overline{X}$ represents an equilibrium point of system (1.9) and $\|.\|$ signifies a norm, for instance, the Euclidean norm.*

1. *$\overline{X}$ is said to be stable (or locally stable) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\left\|X_0 - \overline{X}\right\| < \delta$ it follows that $\left\|X_n - \overline{X}\right\| < \varepsilon$, for $n \geq 0$.*

2. *$\overline{X}$ is said to be asymptotically stable (or locally asymptotically stable) if it is stable and if there exists $\gamma > 0$, such that whenever $\left\|X_0 - \overline{X}\right\| < \gamma$ it follows that*

$$X_n \to \overline{X}, \quad n \to +\infty.$$

3. *$\overline{X}$ is said to be globally attractive (similarly globally attractive of basin of attraction $G \subseteq I_1^{k+1} \times I_2^{k+1} \times \ldots \times I_p^{k+1}$), if for each X_0 (similarly for each $X_0 \in G$)*

$$X_n \to \overline{X}, \quad n \to +\infty.$$

4. *$\overline{X}$ is said to be globally asymptotically stable (similarly globally asymptotically stable relative to G) if it is locally stable, and if for each X_0 (similarly for each $X_0 \in G$),*

$$X_n \to \overline{X}, \quad n \to +\infty.$$

5. *$\overline{X}$ is said to be unstable if it lacks local stability.*

Remark 1.1.1 *It is clear that* $\left(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}\right) \in I_1 \times I_2 \times \cdots \times I_p$ *is an equilibrium of* (1.8) *just in case* $\overline{X} = \left(\overline{x^{(1)}}, \overline{x^{(1)}}, \ldots, \overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}, \overline{x^{(p)}}, \ldots, \overline{x^{(p)}}\right) \in I_1^{k+1} \times I_2^{k+1} \times \ldots \times I_p^{k+1}$ *is an equilibrium of* (1.9).

Definition 1.1.13 (Associated linear system)

We call linear system associated with system (1.9) *around*

$\overline{X} = \left(\overline{x^{(1)}}, \overline{x^{(1)}}, \ldots, \overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}, \overline{x^{(p)}}, \ldots, \overline{x^{(p)}}\right)$,

the system

$$X_{n+1} = J_F X_n, \quad n = 0, 1, 2, \ldots$$

where J_F *denotes the Jacobian matrix of* F *around the equilibrium point* $\overline{X}$*, defined as*

$$J_F = \begin{pmatrix}
\frac{\partial f_0^{(1)}}{\partial u_0^{(1)}} & \frac{\partial f_0^{(1)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_k^{(1)}} & \frac{\partial f_0^{(1)}}{\partial u_0^{(2)}} & \frac{\partial f_0^{(1)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_0^{(p)}} & \frac{\partial f_0^{(1)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_k^{(p)}} \\
\frac{\partial f_1^{(1)}}{\partial u_0^{(1)}} & \frac{\partial f_1^{(1)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_k^{(1)}} & \frac{\partial f_1^{(1)}}{\partial u_0^{(2)}} & \frac{\partial f_1^{(1)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_0^{(p)}} & \frac{\partial f_1^{(1)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_k^{(p)}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_k^{(1)}}{\partial u_0^{(1)}} & \frac{\partial f_k^{(1)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_k^{(1)}} & \frac{\partial f_k^{(1)}}{\partial u_0^{(2)}} & \frac{\partial f_k^{(1)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_0^{(p)}} & \frac{\partial f_k^{(1)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_k^{(p)}} \\
\frac{\partial f_0^{(2)}}{\partial u_0^{(1)}} & \frac{\partial f_0^{(2)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_k^{(1)}} & \frac{\partial f_0^{(2)}}{\partial u_0^{(2)}} & \frac{\partial f_0^{(2)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_0^{(p)}} & \frac{\partial f_0^{(2)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_k^{(p)}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_0^{(p)}}{\partial u_0^{(1)}} & \frac{\partial f_0^{(p)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_k^{(1)}} & \frac{\partial f_0^{(p)}}{\partial u_0^{(2)}} & \frac{\partial f_0^{(p)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_0^{(p)}} & \frac{\partial f_0^{(p)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_k^{(p)}} \\
\frac{\partial f_1^{(p)}}{\partial u_0^{(1)}} & \frac{\partial f_1^{(p)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_k^{(1)}} & \frac{\partial f_1^{(p)}}{\partial u_0^{(2)}} & \frac{\partial f_1^{(p)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_0^{(p)}} & \frac{\partial f_1^{(p)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_k^{(p)}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_k^{(p)}}{\partial u_0^{(1)}} & \frac{\partial f_k^{(p)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_k^{(1)}} & \frac{\partial f_k^{(p)}}{\partial u_0^{(2)}} & \frac{\partial f_k^{(p)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_0^{(p)}} & \frac{\partial f_k^{(p)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_k^{(p)}}
\end{pmatrix},$$

such that

$$f_j^{(i)} = f_j^{(i)}\left(\overline{X}\right), \quad i = \overline{1,p}, \quad j = \overline{0,k}.$$

Theorem 1.1.9 [41] **(Stability by linearization)**

1. *If every eigenvalue of J_F lies within the open unit disk $|\lambda| < 1$, in that case $\overline{X}$ is locally asymptotically stable.*

2. *If at least one of the eigenvalues of J_F has a modulus greater than one, then $\overline{X}$ is unstable.*

Rate of convergence

Here, we are going to give two important propositions which assists in estimating the rate of convergence.

Let's consider the following difference equations system

$$X_{n+1} = (A + B(n)) X_n, \tag{1.10}$$

with X_n represents a vector of dimension m, $A \in C^{m\times m}$ represents a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ represents a matrix function that satisfies the condition

$$\|B(n)\| \to 0 \tag{1.11}$$

as n approaches infinity, where $\|.\|$ signifies any matrix norm corresponding to the vector norm

$$\|(x_1, x_2, \ldots, x_m)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}.$$

Proposition 1.1.1 *[47]* **(The 1^{st} theorem of Perron)**

Suppose condition (1.11) is met. If X_n represents a solution to system (1.10), then either $X_n = 0$ for all sufficiently large n or

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{\frac{1}{n}} \tag{1.12}$$

exists and is equal to the modulus of one of the eigenvalues of A.

Proposition 1.1.2 *[47]* **(The 2^{nd} theorem of Perron)**

Suppose condition (1.11) is met. If X_n represents a solution to system (1.10), then either $X_n = 0$ for all sufficiently large n or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{1.13}$$

exists and is equal to the modulus of one of the eigenvalues of A.

1.2 Solvability of a multidimensional close-to-cyclic system of difference equations

The pursuit of solutions for systems of nonlinear difference equations has sparked significant attention within the academic sphere. However, the majority of the papers published in this aspect were limited to systems of two or three dimensions at most, as evidenced by notable references [4, 5, 16, 17, 19, 24, 25, 26, 29, 30, 31, 32, 33, 35, 36, 37, 38, 40, 51, 55, 56, 60, 62].

The challenges posed by complex calculations and the lack of a straightforward method for solving nonlinear difference equations make it hard for researchers to find direct solutions. As a result, they opt for a different approach: a qualitative study of these systems, where they investigate the periodicity, the local stability, the global stability…(for instance, references such as [12, 15, 21, 22, 23, 27, 28, 34, 41, 43, 46, 48, 61, 63]).

All of the above motivated us to introduce the multidimensional system of nonlinear difference equations (1.14) and solve it, hoping that it will model certain phenomena and help researchers to understand them.

In the second section of this chapter, we are going to extend and refine the findings initially outlined in our publication [6]. So,we are going to find the solutions of the following $k-$dimensional close-to-cyclic nonlinear difference equations system

$$y_{n+1}^{(i)} = \frac{a_i y_n^{(i+1)} \left(y_{n-k}^{(i+1)}\right)^{p_{i+1}} + b_i}{\left(y_{n-k+1}^{(i)}\right)^{p_i}}; \quad n \in \mathbb{N}_0, \tag{1.14}$$

where $y_n^{(i+k)} = y_n^{(i)}, p_{i+k} = p_i, a_{i+k} = a_i, b_{i+k} = b_i; i = \overline{1,k}$, the initial values $y_{-k}^{(i)}, y_{-k+1}^{(i)}, \ldots, y_0^{(i)}$ and the parameters a_i and b_i, $i = \overline{1,k}$ are positive real numbers and p_i, $i = \overline{1,k}$, are real numbers. On top of that, we are going to examine the asymptotic behavior of the equilibrium point of system (1.14) in special cases.

1.2.1 Auxiliary Results

In this part, we are going to present several results needed to prove the main results in part 1.2.2.

Let's examine the following k-dimensional linear difference equations system

$$w_{n+1}^{(i)} = a_i w_n^{(i+1)} + b_i, \quad n \in \mathbb{N}_0 \tag{1.15}$$

where $w_n^{(i+k)} = w_n^{(i)}$ and $w_0^{(i)}, a_i, b_i, i = \overline{1,k}$ are positive real numbers.

The following auxiliary result is used for several times in the rest of the chapter.

Lemma 1.2.1 *Let $(w_n^{(i)})_{n\geq 0}$ be a solution to system (1.15). Then for all $n \in \mathbb{N}_0$*

$$w_{kn+j}^{(i)} = \begin{cases} w_j^{(i)} + nT_i, & S = 1, \\ S^n w_j^{(i)} + T_i\left(\dfrac{S^n - 1}{S - 1}\right), & S \neq 1, \end{cases}$$

where, $i = \overline{1,k}$, $j = \overline{0,k-1}$ and

$$S = \prod_{l=1}^{k} a_l, \quad T_i = \sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2} a_l\right) b_{i+r-1} + b_i. \tag{1.16}$$

Proof. The systems in (1.15) immediately imply, for $i = \overline{1,k}$, the following relations

$$
\begin{aligned}
w_{n+k}^{(i)} &= a_i w_{n+k-1}^{(i+1)} + b_i \\
&= a_i \left[a_{i+1} w_{n+k-2}^{(i+2)} + b_{i+1}\right] + b_i \\
&= a_i a_{i+1} w_{n+k-2}^{(i+2)} + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} \left[a_{i+2} w_{n+k-3}^{(i+3)} + b_{i+2}\right] + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} a_{i+2} w_{n+k-3}^{(i+3)} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} a_{i+2} a_{i+3} w_{n+k-4}^{(i+4)} + a_i a_{i+1} a_{i+2} b_{i+3} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
&\vdots \\
&= a_i a_{i+1} \dots a_{i+k-1} w_{n+k-k}^{(i+k)} + a_i a_{i+1} \dots a_{i+k-2} b_{i+k-1} \\
&+ a_i a_{i+1} \dots a_{i+k-3} b_{i+k-2} + \dots + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} \dots a_{i+k-1} w_n^{(i+k)} + a_i a_{i+1} \dots a_{i+k-2} b_{i+k-1} \\
&+ a_i a_{i+1} \dots a_{i+k-3} b_{i+k-2} + \dots + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} \dots a_{i+k-1} w_n^{(i)} + a_i a_{i+1} \dots a_{i+k-2} b_{i+k-1} \\
&+ a_i a_{i+1} \dots a_{i+k-3} b_{i+k-2} + \dots + a_i b_{i+1} + b_i \\
w_{n+k}^{(i)} &= a_1 a_2 \dots a_k w_n^{(i)} + a_i a_{i+1} \dots a_{i+k-2} b_{i+k-1} \\
&+ a_i a_{i+1} \dots a_{i+k-3} b_{i+k-2} + \dots + a_i b_{i+1} + b_i.
\end{aligned}
$$

So, we have

$$w_{n+k}^{(i)} = \left(\prod_{l=1}^{k} a_l\right) w_n^{(i)} + \left[\sum_{r=2}^{k} \left(\prod_{l=i}^{i+r-2} a_l\right) b_{i+r-1}\right] + b_i.$$

Let's put

$$S = \prod_{l=1}^{k} a_l \ \ and \ \ T_i = \sum_{r=2}^{k} \left(\prod_{l=i}^{i+r-2} a_l\right) b_{i+r-1} + b_i.$$

We get

$$w_{n+k}^{(i)} = S w_n^{(i)} + T_i,$$

for $i = \overline{1,k}$, with the initial values $w_j^{(i)}$, $j = \overline{0, k-1}$.

Consequently, instead of solving system (1.15), we are going to solve the following equations

$$w_{n+k}^{(i)} = Sw_n^{(i)} + T_i, \quad n \in \mathbb{N}_0 \tag{1.17}$$

where, for $i = \overline{1,k}$ and $j = \overline{0,k-1}$, $w_j^{(i)}$ are positive real numbers.

Equations (1.17) yield

$$\begin{aligned}
w_k^{(i)} &= Sw_0^{(i)} + T_i,\\
w_{k+1}^{(i)} &= Sw_1^{(i)} + T_i,\\
&\vdots\\
w_{2k-1}^{(i)} &= Sw_{k-1}^{(i)} + T_i,\\
w_{2k}^{(i)} &= Sw_k^{(i)} + T_i = S\left(Sw_0^{(i)} + T_i\right) + T_i = S^2w_0^{(i)} + ST_i + T_i,\\
w_{2k+1}^{(i)} &= Sw_{k+1}^{(i)} + T_i = S\left(Sw_1^{(i)} + T_i\right) + T_i = S^2w_1^{(i)} + ST_i + T_i,\\
&\vdots\\
w_{3k-1}^{(i)} &= Sw_{2k-1}^{(i)} + T_i = S\left(Sw_{k-1}^{(i)} + T_i\right) + T_i = S^2w_{k-1}^{(i)} + ST_i + T_i,\\
w_{3k}^{(i)} &= Sw_{2k}^{(i)} + T_i = S\left(S^2w_0^{(i)} + ST_i + T_i\right) + T_i = S^3w_0^{(i)} + S^2T_i\\
&+ ST_i + T_i,\\
w_{3k+1}^{(i)} &= Sw_{2k+1}^{(i)} + T_i = S\left(S^2w_1^{(i)} + ST_i + T_i\right) + T_i = S^3w_1^{(i)} + S^2T_i\\
&+ ST_i + T_i,\\
&\vdots\\
w_{4k-1}^{(i)} &= Sw_{3k-1}^{(i)} + T_i = S\left(S^2w_{k-1}^{(i)} + ST_i + T_i\right) + T_i = S^3w_{k-1}^{(i)} + S^2T_i\\
&+ ST_i + T_i.
\end{aligned}$$

The inductive argument proves, for $i = \overline{1,k}$, that

$$\begin{aligned}
w_{kn}^{(i)} &= S^n w_0^{(i)} + \sum_{t=0}^{n-1} S^t T_i,\\
w_{kn+1}^{(i)} &= S^n w_1^{(i)} + \sum_{t=0}^{n-1} S^t T_i,
\end{aligned}$$

$$\begin{aligned} w^{(i)}_{kn+2} &= S^n w^{(i)}_2 + \sum_{t=0}^{n-1} S^t T_i, \\ &\vdots \\ w^{(i)}_{kn+k-1} &= S^n w^{(i)}_{k-1} + \sum_{t=0}^{n-1} S^t T_i. \end{aligned}$$

More precisely, for $i = \overline{1,k}$ and $j = 0, 1, \ldots, k-1$, we obtain

$$w^{(i)}_{kn+j} = S^n w^{(i)}_j + \sum_{t=0}^{n-1} S^t T_i.$$

Thus, for all n$\in \mathbb{N}_0$ we obtain

$$w^{(i)}_{kn+j} = \begin{cases} w^{(i)}_j + nT_i, & S = 1, \\ S^n w^{(i)}_j + T_i\left(\dfrac{S^n - 1}{S-1}\right), & S \neq 1. \end{cases} \tag{1.18}$$

Now, we are going to prove by induction that relation (1.18) is true.

- A simple verification shows that relation (1.18) holds for $n = 0$.

- Suppose that relation (1.18) holds for n, that is

$$w^{(i)}_{kn+j} = \begin{cases} w^{(i)}_j + nT_i, & S = 1, \\ S^n w^{(i)}_j + T_i\left(\dfrac{S^n - 1}{S-1}\right), & S \neq 1. \end{cases}$$

- We are going to prove that relation (1.18) holds for $n+1$. We have

  - If $S \neq 1$

$$\begin{aligned} w^{(i)}_{k(n+1)+j} &= w^{(i)}_{kn+j+k} \\ &= S w^{(i)}_{kn+j} + T_i \\ &= S\left[S^n w^{(i)}_j + T_i\left(\frac{S^n - 1}{S-1}\right)\right] + T_i \\ w^{(i)}_{k(n+1)+j} &= S^{n+1} w^{(i)}_j + T_i\left[S\left(\frac{S^n - 1}{S-1}\right)\right] + T_i. \end{aligned}$$

So

$$\begin{aligned} w^{(i)}_{k(n+1)+j} &= S^{n+1}w^{(i)}_{j} + T_i\left[\frac{S^{n+1}-S}{S-1}\right] + T_i \\ &= S^{n+1}w^{(i)}_{j} + T_i\left[\frac{S^{n+1}-S+S-1}{S-1}\right] \\ w^{(i)}_{k(n+1)+j} &= S^{n+1}w^{(i)}_{j} + T_i\left(\frac{S^{n+1}-1}{S-1}\right). \end{aligned}$$

- If $S = 1$

$$\begin{aligned} w^{(i)}_{k(n+1)+j} &= w^{(i)}_{kn+j+k} \\ &= Sw^{(i)}_{kn+j} + T_i \\ &= w^{(i)}_{kn+j} + T_i \\ &= w^{(i)}_{j} + nT_i + T_i \\ w^{(i)}_{k(n+1)+j} &= w^{(i)}_{j} + (n+1)T_i. \end{aligned}$$

Thus,

$$w^{(i)}_{k(n+1)+j} = \begin{cases} w^{(i)}_{j} + (n+1)\,T_i, & S = 1, \\ S^{n+1}w^{(i)}_{j} + T_i\left(\dfrac{S^{n+1}-1}{S-1}\right), & S \neq 1. \end{cases} \tag{1.19}$$

■

1.2.2 Main results

In this part, we are going to study the solvability of system (1.14) by considering changes of variables which transform it to the system of $k-$linear difference equations (1.15).

Form of solution

Here, we show that the difference equations system (1.14) is practically solvable, and we follow the analysis of each equation of this system. Throughout the paper we will also use the following standard convention:

$$\prod_{j=k}^{k-1} a_j = 1.$$

By using the changes of variables

$$w_n^{(i)} = y_n^{(i)} \left(y_{n-k}^{(i)}\right)^{p_i}, \quad i = \overline{1,k}, \quad n \in \mathbb{N}_0, \tag{1.20}$$

system (1.14) is then converted into the following form

$$w_{n+1}^{(i)} = a_i w_n^{(i+1)} + b_i, \quad i = \overline{1,k}, \quad n \in \mathbb{N}_0$$

which is the same system studied in the previous part.

For $i = \overline{1,k}$, relation (1.20) yield

$$y_n^{(i)} = w_n^{(i)} \left(y_{n-k}^{(i)}\right)^{-p_i}, \quad n \in \mathbb{N}_0.$$

So, for $i = \overline{1,k}$ we get

$$\begin{aligned}
y_{kn}^{(i)} &= w_{kn}^{(i)} \left(y_{kn-k}^{(i)}\right)^{-p_i} \\
&= w_{kn}^{(i)} \left[w_{kn-k}^{(i)} \left(y_{kn-2k}^{(i)}\right)^{-p_i}\right]^{-p_i} \\
&= w_{kn}^{(i)} \left(w_{kn-k}^{(i)}\right)^{-p_i} \left(y_{kn-2k}^{(i)}\right)^{(-p_i)^2} \\
&= w_{kn}^{(i)} \left(w_{kn-k}^{(i)}\right)^{-p_i} \left[w_{kn-2k}^{(i)} \left(y_{kn-3k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^2} \\
&= w_{kn}^{(i)} \left(w_{kn-k}^{(i)}\right)^{-p_i} \left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2} \left(y_{kn-3k}^{(i)}\right)^{(-p_i)^3} \\
&= w_{kn}^{(i)} \left(w_{kn-k}^{(i)}\right)^{-p_i} \left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2} \left[w_{kn-3k}^{(i)} \left(y_{kn-4k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^3} \\
&= w_{kn}^{(i)} \left(w_{kn-k}^{(i)}\right)^{-p_i} \left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn-3k}^{(i)}\right)^{(-p_i)^3} \left(y_{kn-4k}^{(i)}\right)^{(-p_i)^4} \\
&= w_{kn}^{(i)} \left(w_{kn-k}^{(i)}\right)^{(-p_i)^1} \left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2} \ldots \left(w_{kn-(t-1)k}^{(i)}\right)^{(-p_i)^{t-1}} \left(y_{kn-tk}^{(i)}\right)^{(-p_i)^t},
\end{aligned}$$

hence

$$\begin{aligned}
y_{kn}^{(i)} &= w_{kn}^{(i)} \left(w_{kn-k}^{(i)}\right)^{-p_i} \left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn-3k}^{(i)}\right)^{(-p_i)^3} \dots \left(w_{kn-tk}^{(i)}\right)^{(-p_i)^t} \dots \\
&\times \left(w_{k}^{(i)}\right)^{(-p_i)^{n-1}} \left(y_{0}^{(i)}\right)^{(-p_i)^n} \\
&= w_{k(n-0)}^{(i)} \left(w_{k(n-1)}^{(i)}\right)^{-p_i} \left(w_{k(n-2)}^{(i)}\right)^{(-p_i)^2} \left(w_{k(n-3)}^{(i)}\right)^{(-p_i)^3} \dots \left(w_{k(n-t)}^{(i)}\right)^{(-p_i)^t} \dots \\
&\times \left(w_{k(n-(n-1))}^{(i)}\right)^{(-p_i)^{n-1}} \left(y_{0}^{(i)}\right)^{(-p_i)^n}.
\end{aligned}$$

So, we obtain

$$y_{kn}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)}^{(i)}\right)^{(-p_i)^t}\right] \left(y_0^{(i)}\right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \tag{1.21}$$

By the same argument

$$\begin{aligned}
y_{kn+1}^{(i)} &= w_{kn+1}^{(i)} \left(y_{kn+1-k}^{(i)}\right)^{-p_i} \\
&= w_{kn+1}^{(i)} \left[w_{kn+1-k}^{(i)} \left(y_{kn+1-2k}^{(i)}\right)^{-p_i}\right]^{-p_i} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(y_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left[w_{kn+1-2k}^{(i)} \left(y_{kn+1-3k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^2} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \left(y_{kn+1-3k}^{(i)}\right)^{(-p_i)^3} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \left[w_{kn+1-3k}^{(i)} \left(y_{kn+1-4k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^3} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)}\right)^{(-p_i)^3} \left(y_{kn+1-4k}^{(i)}\right)^{(-p_i)^4} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)}\right)^{(-p_i)^3} \dots \\
&\times \left(w_{kn+1-(t-1)k}^{(i)}\right)^{(-p_i)^{t-1}} \left(y_{kn+1-tk}^{(i)}\right)^{(-p_i)^t} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)}\right)^{(-p_i)^3} \dots \\
&\times \left(w_{kn+1-tk}^{(i)}\right)^{(-p_i)^t} \dots \left(w_{k+1}^{(i)}\right)^{(-p_i)^{n-1}} \left(y_{1}^{(i)}\right)^{(-p_i)^n},
\end{aligned}$$

hence

$$\begin{aligned} y^{(i)}_{kn+1} &= w^{(i)}_{k(n-0)+1}\left(w^{(i)}_{k(n-1)+1}\right)^{-p_i}\left(w^{(i)}_{k(n-2)+1}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n-3)+1}\right)^{(-p_i)^3} \\ &\times \left(w^{(i)}_{k(n-t)+1}\right)^{(-p_i)^t}\ldots\left(w^{(i)}_{k(n-(n-1))+1}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_1\right)^{(-p_i)^n}. \end{aligned}$$

So, we get

$$y^{(i)}_{kn+1} = \left[\prod_{t=0}^{n-1}\left(w^{(i)}_{k(n-t)+1}\right)^{(-p_i)^t}\right]\left(y^{(i)}_1\right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \tag{1.22}$$

Likewise

$$\begin{aligned} y^{(i)}_{kn+2} &= w^{(i)}_{kn+2}\left(y^{(i)}_{kn+2-k}\right)^{-p_i} \\ &= w^{(i)}_{kn+2}\left[w^{(i)}_{kn+2-k}\left(y^{(i)}_{kn+2-2k}\right)^{-p_i}\right]^{-p_i} \\ &= w^{(i)}_{kn+2}\left(w^{(i)}_{kn+2-k}\right)^{-p_i}\left(y^{(i)}_{kn+2-2k}\right)^{(-p_i)^2} \\ &= w^{(i)}_{kn+2}\left(w^{(i)}_{kn+2-k}\right)^{-p_i}\left[w^{(i)}_{kn+2-2k}\left(y^{(i)}_{kn+2-3k}\right)^{-p_i}\right]^{(-p_i)^2} \\ &= w^{(i)}_{kn+2}\left(w^{(i)}_{kn+2-k}\right)^{-p_i}\left(w^{(i)}_{kn+2-2k}\right)^{(-p_i)^2}\left[w^{(i)}_{kn+2-3k}\left(y^{(i)}_{kn+2-4k}\right)^{-p_i}\right]^{(-p_i)^3} \\ &= w^{(i)}_{kn+2}\left(w^{(i)}_{kn+2-k}\right)^{-p_i}\left(w^{(i)}_{kn+2-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+2-3k}\right)^{(-p_i)^3}\left(y^{(i)}_{kn+2-4k}\right)^{(-p_i)^4} \\ &= w^{(i)}_{kn+2}\left(w^{(i)}_{kn+2-k}\right)^{-p_i}\left(w^{(i)}_{kn+2-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+2-3k}\right)^{(-p_i)^3}\ldots \\ &\times \left(w^{(i)}_{kn+2-(t-1)k}\right)^{(-p_i)^{t-1}}\left(y^{(i)}_{kn+2-tk}\right)^{(-p_i)^t} \\ &= w^{(i)}_{kn+2}\left(w^{(i)}_{kn+2-k}\right)^{-p_i}\left(w^{(i)}_{kn+2-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+2-3k}\right)^{(-p_i)^3} \\ &\times \ldots\left(w^{(i)}_{kn+2-tk}\right)^{(-p_i)^t}\ldots\left(w^{(i)}_{k+2}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_2\right)^{(-p_i)^n} \\ y^{(i)}_{kn+2} &= w^{(i)}_{k(n-0)+2}\left(w^{(i)}_{k(n-1)+2}\right)^{-p_i}\left(w^{(i)}_{k(n-2)+2}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n-3)+2}\right)^{(-p_i)^3}\ldots \\ &\times \left(w^{(i)}_{k(n-t)+2}\right)^{(-p_i)^t}\ldots\left(w^{(i)}_{k(n-(n-1))+2}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_2\right)^{(-p_i)^n}. \end{aligned}$$

So, we get

$$y^{(i)}_{kn+2} = \left[\prod_{t=0}^{n-1}\left(w^{(i)}_{k(n-t)+2}\right)^{(-p_i)^t}\right]\left(y^{(i)}_2\right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \tag{1.23}$$

By the same argument

$$\begin{aligned} y^{(i)}_{kn+k-1} &= w^{(i)}_{kn+k-1}\left(y^{(i)}_{kn+k-1-k}\right)^{-p_i} \\ &= w^{(i)}_{kn+k-1}\left[w^{(i)}_{kn+k-1-k}\left(y^{(i)}_{kn+k-1-2k}\right)^{-p_i}\right]^{-p_i} \\ y^{(i)}_{kn+k-1} &= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(y^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2}. \end{aligned}$$

Hence

$$\begin{aligned} y^{(i)}_{kn+k-1} &= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left[w^{(i)}_{kn+k-1-2k}\left(y^{(i)}_{kn+k-1-3k}\right)^{-p_i}\right]^{(-p_i)^2} \\ &= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2} \\ &\times \left[w^{(i)}_{kn+k-1-3k}\left(y^{(i)}_{kn+k-1-4k}\right)^{-p_i}\right]^{(-p_i)^3} \\ &= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+k-1-3k}\right)^{(-p_i)^3} \\ &\times \left(y^{(i)}_{kn+k-1-4k}\right)^{(-p_i)^4} \\ &= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+k-1-3k}\right)^{(-p_i)^3} \\ &\times \ldots\left(w^{(i)}_{kn+k-1-(t-1)k}\right)^{(-p_i)^{t-1}}\left(y^{(i)}_{kn+k-1-tk}\right)^{(-p_i)^t} \\ &= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+k-1-3k}\right)^{(-p_i)^3} \\ &\times \ldots\left(w^{(i)}_{kn+k-1-tk}\right)^{(-p_i)^t}\ldots\left(w^{(i)}_{2k-1}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_{k-1}\right)^{(-p_i)^n} \\ y^{(i)}_{kn+k-1} &= w^{(i)}_{k(n-0)+k-1}\left(w^{(i)}_{k(n-1)+k-1}\right)^{-p_i}\left(w^{(i)}_{k(n-2)+k-1}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n-3)+k-1}\right)^{(-p_i)^3} \\ &\times \ldots\left(w^{(i)}_{k(n-t)+k-1}\right)^{(-p_i)^t}\ldots\left(w^{(i)}_{k(n-(n-1))+k-1}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_{k-1}\right)^{(-p_i)^n}. \end{aligned}$$

So, we get

$$y^{(i)}_{kn+k-1} = \left[\prod_{t=0}^{n-1}\left(w^{(i)}_{k(n-t)+k-1}\right)^{(-p_i)^t}\right]\left(y^{(i)}_{k-1}\right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \tag{1.24}$$

From (1.21), (1.22), (1.23) and (1.24), we can deduce that for $i = \overline{1,k}$ and $j = \overline{0,k-1}$, we obtain

$$y^{(i)}_{kn+j} = \left[\prod_{t=0}^{n-1}\left(w^{(i)}_{k(n-t)+j}\right)^{(-p_i)^t}\right]\left(y^{(i)}_j\right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \tag{1.25}$$

Now, we are going to prove by induction that relation (1.25) is true.

•A simple verification shows that relation (1.25) holds for $n = 0$.

• Assume that relation (1.25) holds for n, that is

$$y^{(i)}_{kn+j} = \left[\prod_{t=0}^{n-1}\left(w^{(i)}_{k(n-t)+j}\right)^{(-p_i)^t}\right]\left(y^{(i)}_j\right)^{(-p_i)^n}.$$

• We are going to prove that relation (1.25) holds for $n+1$. We get

$$\begin{aligned}
y^{(i)}_{k(n+1)+j} &= w^{(i)}_{k(n+1)+j}\left(y^{(i)}_{k(n+1)+j-k}\right)^{-p_i}\\
&= w^{(i)}_{k(n+1)+j}\left[w^{(i)}_{k(n+1)+j-k}\left(y^{(i)}_{k(n+1)+j-2k}\right)^{-p_i}\right]^{-p_i}\\
&= w^{(i)}_{k(n+1)+j}\left(w^{(i)}_{k(n+1)+j-k}\right)^{-p_i}\left(y^{(i)}_{k(n+1)+j-2k}\right)^{(-p_i)^2}\\
&= w^{(i)}_{k(n+1)+j}\left(w^{(i)}_{k(n+1)+j-k}\right)^{-p_i}\left[w^{(i)}_{k(n+1)+j-2k}\left(y^{(i)}_{k(n+1)+j-3k}\right)^{-p_i}\right]^{(-p_i)^2}\\
&= w^{(i)}_{k(n+1)+j}\left(w^{(i)}_{k(n+1)+j-k}\right)^{-p_i}\left(w^{(i)}_{k(n+1)+j-2k}\right)^{(-p_i)^2}\\
&\times \left[w^{(i)}_{k(n+1)+j-3k}\left(y^{(i)}_{k(n+1)+j-4k}\right)^{-p_i}\right]^{(-p_i)^3}\\
&= w^{(i)}_{k(n+1)+j}\left(w^{(i)}_{k(n+1)+j-k}\right)^{-p_i}\left(w^{(i)}_{k(n+1)+j-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n+1)+j-3k}\right)^{(-p_i)^3}\\
&\times \left(y^{(i)}_{k(n+1)+j-4k}\right)^{(-p_i)^4}\\
&= w^{(i)}_{k(n+1)+j}\left(w^{(i)}_{k(n+1)+j-k}\right)^{-p_i}\left(w^{(i)}_{k(n+1)+j-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n+1)+j-3k}\right)^{(-p_i)^3}\\
&\times \ldots\left(w^{(i)}_{k(n+1)+j-(t-1)k}\right)^{(-p_i)^{t-1}}\left(y^{(i)}_{k(n+1)+j-tk}\right)^{(-p_i)^t}\\
y^{(i)}_{k(n+1)+j} &= w^{(i)}_{k(n+1)+j}\left(w^{(i)}_{k(n+1)+j-k}\right)^{-p_i}\left(w^{(i)}_{k(n+1)+j-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n+1)+j-3k}\right)^{(-p_i)^3}\\
&\times \ldots\left(w^{(i)}_{k(n+1)+j-tk}\right)^{(-p_i)^t}\ldots\left(w^{(i)}_{k+j}\right)^{(-p_i)^n}\left(y^{(i)}_j\right)^{(-p_i)^{n+1}},
\end{aligned}$$

hence

$$\begin{aligned} y^{(i)}_{k(n+1)+j} &= w^{(i)}_{k(n+1-0)+j}\left(w^{(i)}_{k(n+1-1)+j}\right)^{-p_i}\left(w^{(i)}_{k(n+1-2)+j}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n+1-3)+j}\right)^{(-p_i)^3} \\ &\times \ldots\left(w^{(i)}_{k(n+1-t)+j}\right)^{(-p_i)^t}\ldots\left(w^{(i)}_{k(n+1-n)+j}\right)^{(-p_i)^n}\left(y^{(i)}_j\right)^{(-p_i)^{n+1}}. \end{aligned}$$

So,

$$y^{(i)}_{k(n+1)+j} = \left[\prod_{t=0}^{n}\left(w^{(i)}_{k(n+1-t)+j}\right)^{(-p_i)^t}\right]\left(y^{(i)}_j\right)^{(-p_i)^{n+1}}.$$

The results below provide a precise formula for the solution of system (1.14).

Theorem 1.2.1 *Suppose* $\{y^{(i)}_n\}_{n\geq -k}$ *represents a well defined solution of system* (1.14)*. Then, for* $i=\overline{1,k}$, $j=\overline{0,k-1}$ *and* $n\in\mathbb{N}_0$, *we have*

- *If* $S\neq 1$

$$y^{(i)}_{kn+j} = \left[\prod_{t=0}^{n-1}\left(S^{n-t}y^{(i)}_j\left(y^{(i)}_{j-k}\right)^{p_i}+T_i\left(\frac{S^{n-t}-1}{S-1}\right)\right)^{(-p_i)^t}\right]\left(y^{(i)}_j\right)^{(-p_i)^n}.$$

- *If* $S=1$

$$y^{(i)}_{kn+j} = \left[\prod_{t=0}^{n-1}\left(y^{(i)}_j\left(y^{(i)}_{j-k}\right)^{p_i}+(n-t)T_i\right)^{(-p_i)^t}\right]\left(y^{(i)}_j\right)^{(-p_i)^n}.$$

Asymptotic behavior

Here, we are going to study the asymptotic behavior of the equilibrium point of system (1.14).

The following lemma gives the equilibrium of system (1.14).

Lemma 1.2.2

If $\left(\overline{y^{(1)}}, \overline{y^{(2)}}, \ldots, \overline{y^{(k-1)}}, \overline{y^{(k)}}\right)$ *is an equilibrium point of system* (1.14)*, then it is given by*

$$\left(\left[\frac{T_1}{1-S}\right]^{\frac{1}{p_1+1}}, \left[\frac{T_2}{1-S}\right]^{\frac{1}{p_2+1}}, \ldots, \left[\frac{T_{k-1}}{1-S}\right]^{\frac{1}{p_{k-1}+1}}, \left[\frac{T_k}{1-S}\right]^{\frac{1}{p_k+1}}\right),$$

with $S = \prod_{l=1}^{k} a_l < 1$.

Proof. Let $\left(\overline{y^{(1)}}, \overline{y^{(2)}}, \ldots, \overline{y^{(k-1)}}, \overline{y^{(k)}}\right)$ be an equilibrium point of system (1.14). So, from system (1.14) and for $i = \overline{1,k}$ we have

$$\begin{aligned}
\left(\overline{y^{(i)}}\right)^{p_i+1} &= a_i \left(\overline{y^{(i+1)}}\right)^{p_{i+1}+1} + b_i \\
&= a_i \left[a_{i+1} \left(\overline{y^{(i+2)}}\right)^{p_{i+2}+1} + b_{i+1}\right] + b_i \\
&= a_i a_{i+1} \left(\overline{y^{(i+2)}}\right)^{p_{i+2}+1} + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} \left[a_{i+2} \left(\overline{y^{(i+3)}}\right)^{p_{i+3}+1} + b_{i+2}\right] + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} a_{i+2} \left(\overline{y^{(i+3)}}\right)^{p_{i+3}+1} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} a_{i+2} a_{i+3} \left(\overline{y^{(i+4)}}\right)^{p_{i+4}+1} + a_i a_{i+1} a_{i+2} b_{i+3} \\
&+ a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} a_{i+2} a_{i+3} a_{i+4} \left(\overline{y^{(i+5)}}\right)^{p_{i+5}+1} + a_i a_{i+1} a_{i+2} a_{i+3} b_{i+4} \\
&+ a_i a_{i+1} a_{i+2} b_{i+3} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
&\vdots \\
&= a_i a_{i+1} \ldots a_{i+k-1} \left(\overline{y^{(i+k)}}\right)^{p_{i+k}+1} + a_i a_{i+1} \ldots a_{i+k-2} b_{i+k-1} \\
&+ a_i a_{i+1} \ldots a_{i+k-3} b_{i+k-2} + \ldots + a_i b_{i+1} + b_i \\
\left(\overline{y^{(i)}}\right)^{p_i+1} &= a_i a_{i+1} \ldots a_{i+k-1} \left(\overline{y^{(i)}}\right)^{p_i+1} + a_i a_{i+1} \ldots a_{i+k-2} b_{i+k-1} \\
&+ a_i a_{i+1} \ldots a_{i+k-3} b_{i+k-2} + \ldots + a_i b_{i+1} + b_i.
\end{aligned}$$

Hence

$$\begin{aligned}\left(\overline{y^{(i)}}\right)^{p_i+1} &= a_1a_2\dots a_k\left(\overline{y^{(i)}}\right)^{p_i+1} + a_ia_{i+1}\dots a_{i+k-2}b_{i+k-1}\\ &+ a_ia_{i+1}\dots a_{i+k-3}b_{i+k-2}+\dots+a_ib_{i+1}+b_i\\ &= \left(\prod_{l=1}^{k}a_l\right)\left(\overline{y^{(i)}}\right)^{p_i+1} + \left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i.\end{aligned}$$

So

$$\left(\overline{y^{(i)}}\right)^{p_i+1}\left(1-\prod_{l=1}^{k}a_l\right) = \left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i,$$

consequently

$$\overline{y^{(i)}} = \left[\frac{\left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i}{1-\prod_{l=1}^{k}a_l}\right]^{\dfrac{1}{p_i+1}}.$$

Using notation (1.16), we get

$$\overline{y^{(i)}} = \left[\frac{T_i}{1-S}\right]^{\frac{1}{p_i+1}}, \quad i=\overline{1,k}.$$

Note that the condition $S<1$ implies that $\overline{y^{(i)}}$ is positive whatever the values of p_i, $i=\overline{1,k}$. ■

Theorem 1.2.2 *Consider system (1.14). Assume, for $i=\overline{1,k}$, that $S<1$ and $\mid p_i \mid< 1$. Then, the equilibrium point of system (1.14) is globally attractive.*

Proof. Suppose, for $i=\overline{1,k}$, that $S<1$ and $\mid p_i\mid<1$, so we obtain

$$\lim_{n\to+\infty} y^{(i)}_{kn+j} = \lim_{n\to+\infty}\left[\left(\prod_{t=0}^{n-1}\left(S^{n-t}y^{(i)}_j\left(y^{(i)}_{j-k}\right)^{p_i}+T_i\left(\frac{S^{n-t}-1}{S-1}\right)\right)^{(-p_i)^t}\right)\left(y^{(i)}_j\right)^{(-p_i)^n}\right].$$

Hence

$$\begin{aligned}
\lim_{n\to+\infty} y_{kn+j}^{(i)} &= \prod_{t\geq 0}\left[T_i\left(\frac{-1}{S-1}\right)\right]^{(-p_i)^t} \\
&= \prod_{t\geq 0}\left[\frac{T_i}{1-S}\right]^{(-p_i)^t} \\
\lim_{n\to+\infty} y_{kn+j}^{(i)} &= \left[\frac{T_i}{1-S}\right]^{\sum_{t\geq 0}(-p_i)^t}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\sum_{t\geq 0}(-p_i)^t &= \lim_{m\to+\infty}\sum_{t=0}^{m}(-p_i)^t \\
&= \lim_{m\to+\infty}\frac{(-p_i)^{m+1}-1}{-p_i-1} \\
&= \frac{-1}{-p_i-1} \\
\sum_{t\geq 0}(-p_i)^t &= \frac{1}{p_i+1}.
\end{aligned}$$

So

$$\lim_{n\to+\infty} y_{kn+j}^{(i)} = \left[\frac{T_i}{1-S}\right]^{\frac{1}{p_i+1}} = \overline{y^{(i)}}.$$

From where the equilibrium is globally attractive.

■

1.2.3 Numerical examples

Example 1.2.1 *Let $k = 2$, $a_1 = 2, a_2 = \frac{1}{2}, b_1 = 2, b_2 = 3, p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{3}$, and the initial values*

$$y_{-2}^{(1)} = 4,\ \ y_{-1}^{(1)} = 4,\ \ y_{0}^{(1)} = 3,\ \ y_{-2}^{(2)} = 8,\ \ y_{-1}^{(2)} = 8 \ \text{ and } \ y_{0}^{(2)} = 6 \tag{1.26}$$

in system (1.14), then we obtain that $S = 1$, and for $i = 1, 2$, we have $| p_i |< 1$. So we obtain the following system

$$y_{n+1}^{(1)} = \frac{2y_n^{(2)}\left(y_{n-2}^{(2)}\right)^{\frac{1}{3}}+2}{\left(y_{n-1}^{(1)}\right)^{\frac{1}{2}}}, \quad y_{n+1}^{(2)} = \frac{\frac{1}{2}y_n^{(1)}\left(y_{n-2}^{(1)}\right)^{\frac{1}{2}}+3}{\left(y_{n-1}^{(2)}\right)^{\frac{1}{3}}}, \quad n \in \mathbb{N}_0. \tag{1.27}$$

The solution of system (1.27) is given by

$$\begin{aligned}
y_{2n}^{(1)} &= \left[\prod_{t=0}^{n-1}(6+8(n-t))^{\left(-\frac{1}{2}\right)^t}\right] \\
&\times (3)^{\left(-\frac{1}{2}\right)^n}, \\
y_{2n+1}^{(1)} &= \left[\prod_{t=0}^{n-1}(26+8(n-t))^{\left(-\frac{1}{2}\right)^t}\right] \\
&\times (13)^{\left(-\frac{1}{2}\right)^n}, \\
y_{2n}^{(2)} &= \left[\prod_{t=0}^{n-1}(12+4(n-t))^{\left(-\frac{1}{3}\right)^t}\right] \\
&\times (6)^{\left(-\frac{1}{3}\right)^n}, \\
y_{2n+1}^{(2)} &= \left[\prod_{t=0}^{n-1}(6+4(n-t))^{\left(-\frac{1}{3}\right)^t}\right] \\
&\times (3)^{\left(-\frac{1}{3}\right)^n},
\end{aligned}$$

for all $n \geq 0$.

The behavior of the solution of system (1.27) is represented in figure (1.1).

Example 1.2.2 *Let $k = 2$, $a_1 = 1, a_2 = \frac{2}{3}, b_1 = 2, b_2 = 3, p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{3}$, and the initial values*

$$y_{-2}^{(1)} = 4, \;\; y_{-1}^{(1)} = 4, \;\; y_0^{(1)} = 3, \;\; y_{-2}^{(2)} = 8, \;\; y_{-1}^{(2)} = 8 \;\text{ and }\; y_0^{(2)} = 6 \tag{1.28}$$

in system (1.14), then we obtain that $S < 1$, and for $i = 1, 2$, we have $\mid p_i \mid < 1$. So we obtain the following system

$$y_{n+1}^{(1)} = \frac{y_n^{(2)}\left(y_{n-2}^{(2)}\right)^{\frac{1}{3}}+2}{\left(y_{n-1}^{(1)}\right)^{\frac{1}{2}}}, \quad y_{n+1}^{(2)} = \frac{\frac{2}{3}y_n^{(1)}\left(y_{n-2}^{(1)}\right)^{\frac{1}{2}}+3}{\left(y_{n-1}^{(2)}\right)^{\frac{1}{3}}}, \quad n \in \mathbb{N}_0. \tag{1.29}$$

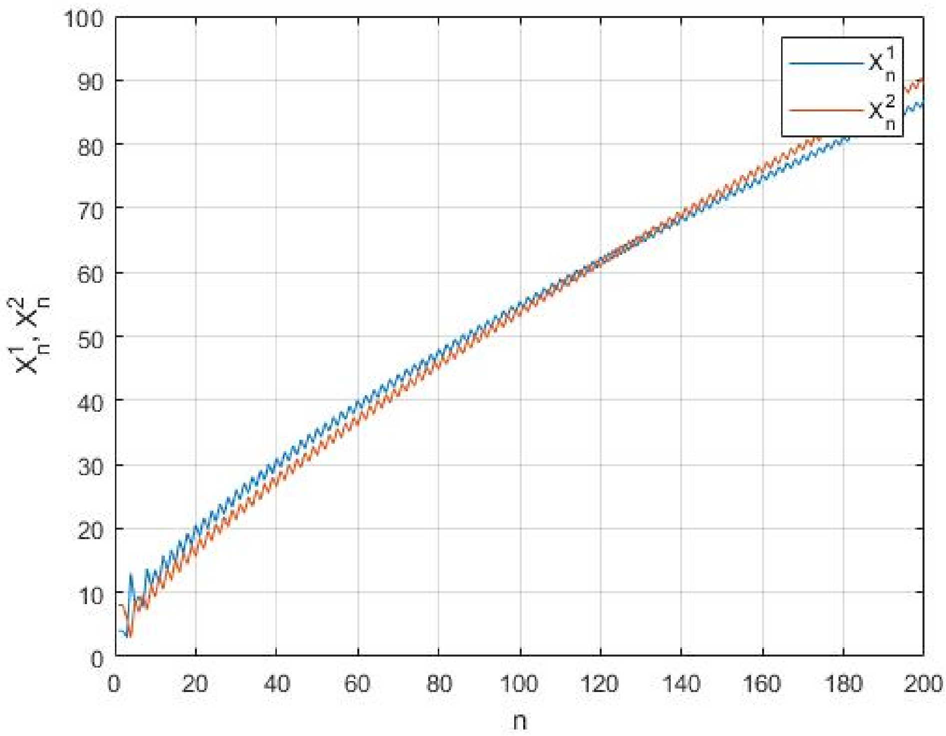

Figure 1.1: Plot of system (1.27) using the initial values (1.26).

The solution of system (1.29) *is given by*

$$
\begin{aligned}
y^{(1)}_{2n} &= \left[\prod_{t=0}^{n-1}\left(6\left(\frac{2}{3}\right)^{n-t}+15\left(1-\left(\frac{2}{3}\right)^{n-t}\right)\right)^{\left(-\frac{1}{2}\right)^t}\right] \\
&\times\ (3)^{\left(-\frac{1}{2}\right)^n}, \\
y^{(1)}_{2n+1} &= \left[\prod_{t=0}^{n-1}\left(14\left(\frac{2}{3}\right)^{n-t}+15\left(1-\left(\frac{2}{3}\right)^{n-t}\right)\right)^{\left(-\frac{1}{2}\right)^t}\right] \\
&\times\ (7)^{\left(-\frac{1}{2}\right)^n}, \\
y^{(2)}_{2n} &= \left[\prod_{t=0}^{n-1}\left(12\left(\frac{2}{3}\right)^{n-t}+13\left(1-\left(\frac{2}{3}\right)^{n-t}\right)\right)^{\left(-\frac{1}{3}\right)^t}\right] \\
&\times\ (6)^{\left(-\frac{1}{3}\right)^n},
\end{aligned}
$$

$$
\begin{aligned}
y^{(2)}_{2n+1} &= \left[\prod_{t=0}^{n-1}\left(7\left(\frac{2}{3}\right)^{n-t}+13\left(1-\left(\frac{2}{3}\right)^{n-t}\right)\right)^{\left(-\frac{1}{3}\right)^{t}}\right] \\
&\times \left(\frac{7}{2}\right)^{\left(-\frac{1}{3}\right)^{n}},
\end{aligned}
$$

for all $n \geq 0$.

The solution of system (1.29) *converges to the equilibrium point* $(\overline{y^{(1)}}, \overline{y^{(2)}}) = \left(15^{\frac{2}{3}}, 13^{\frac{3}{4}}\right)$ *(see Figure* (1.2)*, Theorem* (1.2.2)*).*

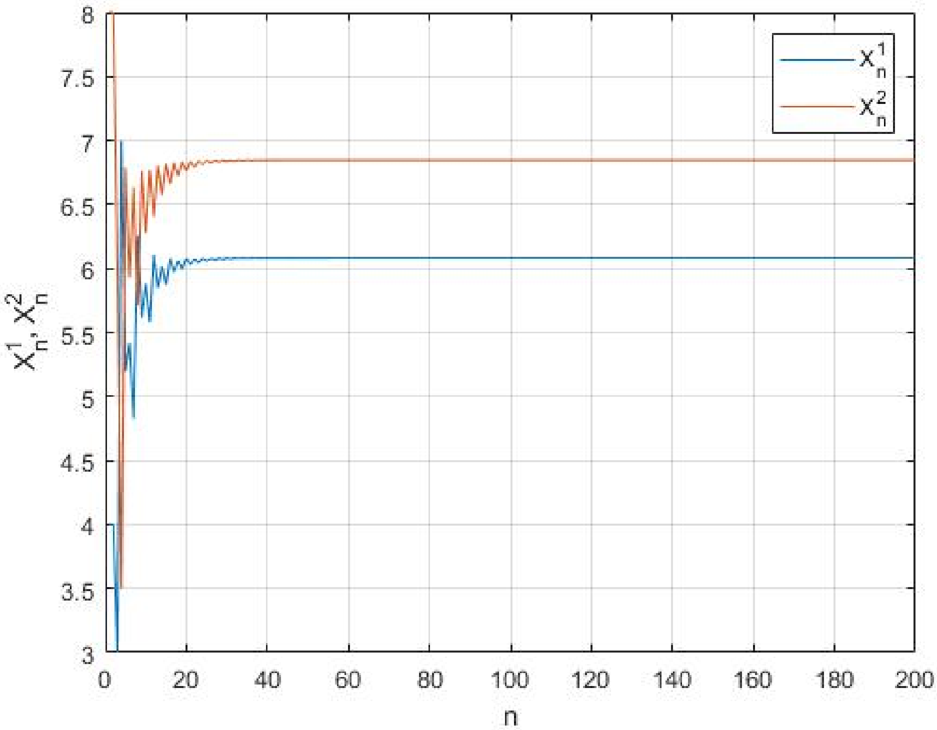

Figure 1.2: Plot of system (1.29) using the initial values (1.28).

Example 1.2.3 *Let* $k = 6$, $a_i = b_i = \frac{1}{2}$, *for* $i = 1, 2, \ldots, 6$ *and* $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$, $p_3 = \frac{3}{5}$, $p_4 = \frac{9}{10}$, $p_5 = \frac{-7}{10}$ *and* $p_6 = \frac{4}{5}$

in system (1.14), *then we obtain that* $S < 1$, *and for* $i = 1, 2, \ldots 6$, *we have* $\mid p_i \mid < 1$. *So we obtain the following system*

$$\begin{array}{l} y_{n+1}^{(1)} = \dfrac{\frac{1}{2} y_n^{(2)} \left(y_{n-2}^{(2)}\right)^{\frac{1}{2}} + \frac{1}{2}}{\left(y_{n-1}^{(1)}\right)^{\frac{1}{2}}} \\ y_{n+1}^{(2)} = \dfrac{\frac{1}{2} y_n^{(3)} \left(y_{n-2}^{(3)}\right)^{\frac{3}{5}} + \frac{1}{2}}{\left(y_{n-1}^{(2)}\right)^{\frac{1}{2}}} \\ y_{n+1}^{(3)} = \dfrac{\frac{1}{2} y_n^{(4)} \left(y_{n-2}^{(4)}\right)^{\frac{9}{10}} + \frac{1}{2}}{\left(y_{n-1}^{(3)}\right)^{\frac{3}{5}}} \\ y_{n+1}^{(4)} = \dfrac{\frac{1}{2} y_n^{(5)} \left(y_{n-2}^{(5)}\right)^{\frac{-7}{10}} + \frac{1}{2}}{\left(y_{n-1}^{(4)}\right)^{\frac{9}{10}}} \\ y_{n+1}^{(5)} = \dfrac{\frac{1}{2} y_n^{(6)} \left(y_{n-2}^{(6)}\right)^{\frac{4}{5}} + \frac{1}{2}}{\left(y_{n-1}^{(5)}\right)^{\frac{-7}{10}}} \\ y_{n+1}^{(6)} = \dfrac{\frac{1}{2} y_n^{(1)} \left(y_{n-2}^{(1)}\right)^{\frac{1}{2}} + \frac{1}{2}}{\left(y_{n-1}^{(6)}\right)^{\frac{4}{5}}} \end{array}, \quad n \in \mathbb{N}_0, \tag{1.30}$$

with the following initial values

$$\begin{array}{l} y_{-2}^{(1)} = 1,\ y_{-1}^{(1)} = 2,\ y_0^{(1)} = 3,\ y_{-2}^{(2)} = 4,\ y_{-1}^{(2)} = 5,\ y_0^{(2)} = 6,\ y_{-2}^{(3)} = 2,\ y_{-1}^{(3)} = 3,\ y_0^{(3)} = 1, \\ y_{-2}^{(4)} = 2,\ y_{-1}^{(4)} = 2,\ y_0^{(4)} = 3,\ y_{-2}^{(5)} = 3,\ y_{-1}^{(5)} = 2,\ y_0^{(5)} = 2,\ y_{-2}^{(6)} = 4,\ y_{-1}^{(6)} = 2 \text{ and } y_0^{(6)} = 3, \end{array} \tag{1.31}$$

the solution of system (1.30) *converges to the equilibrium point* $(\overline{y^{(1)}}, \overline{y^{(2)}}, \overline{y^{(3)}}, \overline{y^{(4)}}, \overline{y^{(5)}}, \overline{y^{(6)}}) = (1, 1, 1, 1, 1, 1)$ *(see Figure* (1.3), *Theorem* (1.2.2)*).*

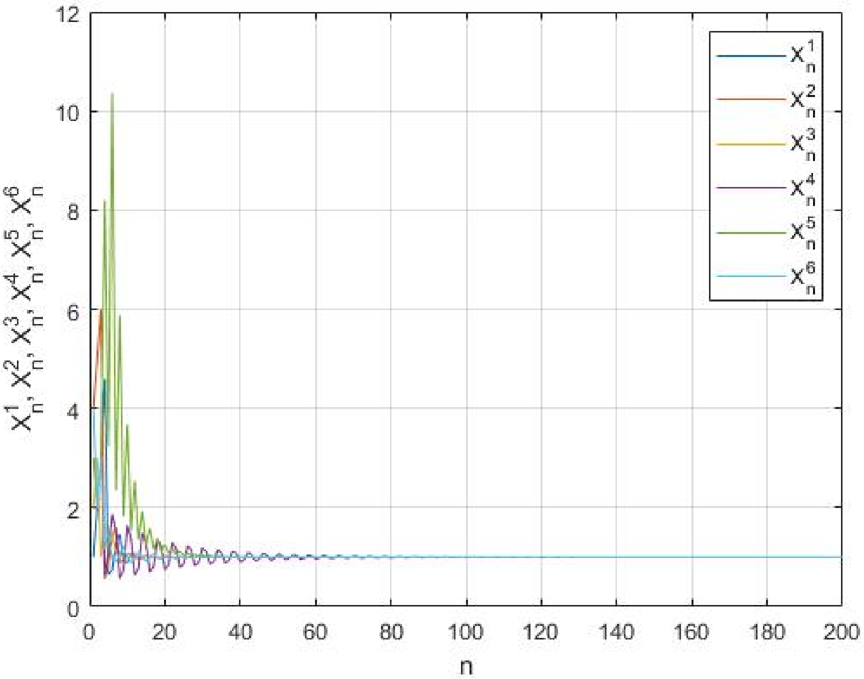

Figure 1.3: Plot of system (1.30) using the initial values (1.31).

Example 1.2.4 *Let $k = 4$, $a_i = b_i = \dfrac{1}{2}$ and $p_i = 1$, for $i = 1, 2, 3, 4$, and the initial values*

$$\begin{aligned} &y_{-2}^{(1)} = 1,\ \ y_{-1}^{(1)} = 2,\ \ y_{0}^{(1)} = 3,\ \ y_{-2}^{(2)} = 4,\ \ y_{-1}^{(2)} = 5,\ \ y_{0}^{(2)} = 6, \\ &y_{-2}^{(3)} = 2,\ \ y_{-1}^{(3)} = 3,\ \ y_{0}^{(3)} = 1,\ \ y_{-2}^{(4)} = 2,\ \ y_{-1}^{(4)} = 2 \ \ and\ \ y_{0}^{(4)} = 3 \end{aligned} \tag{1.32}$$

in system (1.14), then we obtain that $S < 1$, and for $i = 1, 2, 3, 4$, we have $\mid p_i \mid= 1$. So we obtain the following system

$$\left.\begin{aligned} y_{n+1}^{(1)} &= \frac{\frac{1}{2}y_n^{(2)}y_{n-2}^{(2)} + \frac{1}{2}}{y_{n-1}^{(1)}} \\ y_{n+1}^{(2)} &= \frac{\frac{1}{2}y_n^{(3)}y_{n-2}^{(3)} + \frac{1}{2}}{y_{n-1}^{(2)}} \\ y_{n+1}^{(3)} &= \frac{\frac{1}{2}y_n^{(4)}y_{n-2}^{(4)} + \frac{1}{2}}{y_{n-1}^{(3)}} \\ y_{n+1}^{(4)} &= \frac{\frac{1}{2}y_n^{(1)}y_{n-2}^{(1)} + \frac{1}{2}}{y_{n-1}^{(4)}} \end{aligned}\right\} ,\ \ n \in \mathbb{N}_0. \tag{1.33}$$

The equilibrium $(\overline{y^{(1)}}, \overline{y^{(2)}}, \overline{y^{(3)}}, \overline{y^{(4)}})$ *is not globally attractive (see Figure* (1.4)*, Theorem* (1.2.2)*).*

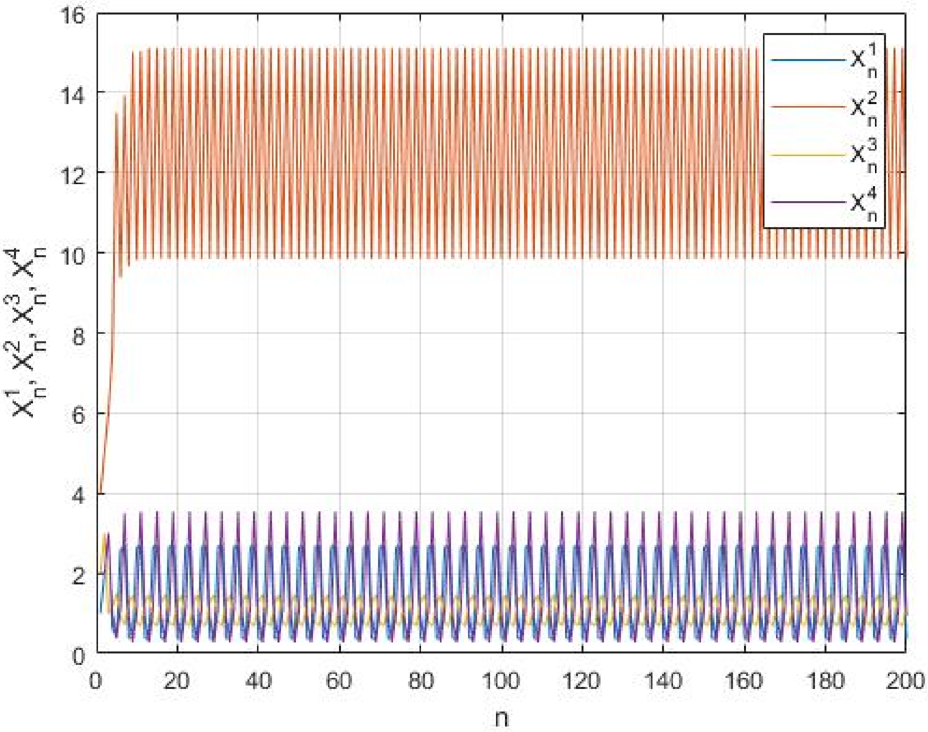

Figure 1.4: Plot of system (1.33) using the initial values (1.32).

Example 1.2.5 *Let* $k = 4$*,* $a_i = b_i = \dfrac{1}{2}$ *, for* $i = 1, 2, 3, 4$*, and* $p_1 = \dfrac{3}{2}$*,* $p_2 = \dfrac{-9}{5}$*,* $p_3 = \dfrac{31}{10}$*,* $p_4 = \dfrac{51}{10}$*, and the initial values*

$$\begin{aligned} &y^{(1)}_{-2} = 1,\ \ y^{(1)}_{-1} = 2,\ \ y^{(1)}_{0} = 3,\ \ y^{(2)}_{-2} = 4,\ \ y^{(2)}_{-1} = 5,\ \ y^{(2)}_{0} = 6,\\ &y^{(3)}_{-2} = 2,\ \ y^{(3)}_{-1} = 3,\ \ y^{(3)}_{0} = 1,\ \ y^{(4)}_{-2} = 2,\ \ y^{(4)}_{-1} = 2\ \ \text{and}\ \ y^{(4)}_{0} = 3 \end{aligned} \tag{1.34}$$

in system (1.14)*, then we obtain that* $S < 1$*, and for* $i = 1, 2, 3, 4$*, we have* $|\, p_i \,| > 1$*. So we obtain the following system*

$$\left.\begin{aligned} y_{n+1}^{(1)} &= \frac{\frac{1}{2}y_n^{(2)}\left(y_{n-2}^{(2)}\right)^{\frac{-9}{5}}+\frac{1}{2}}{\left(y_{n-1}^{(1)}\right)^{\frac{3}{2}}} \\ y_{n+1}^{(2)} &= \frac{\frac{1}{2}y_n^{(3)}\left(y_{n-2}^{(3)}\right)^{\frac{31}{10}}+\frac{1}{2}}{\left(y_{n-1}^{(2)}\right)^{\frac{-9}{10}}} \\ y_{n+1}^{(3)} &= \frac{\frac{1}{2}y_n^{(4)}\left(y_{n-2}^{(4)}\right)^{\frac{51}{10}}+\frac{1}{2}}{\left(y_{n-1}^{(3)}\right)^{\frac{31}{10}}} \\ y_{n+1}^{(4)} &= \frac{\frac{1}{2}y_n^{(5)}\left(y_{n-2}^{(1)}\right)^{\frac{3}{2}}+\frac{1}{2}}{\left(y_{n-1}^{(4)}\right)^{\frac{51}{10}}} \end{aligned}\right., \quad n \in \mathbb{N}_0. \tag{1.35}$$

The equilibrium $(\overline{y^{(1)}}, \overline{y^{(2)}}, \overline{y^{(3)}}, \overline{y^{(4)}})$ *is not globally attractive (see Figure (1.5), Theorem (1.2.2)).*

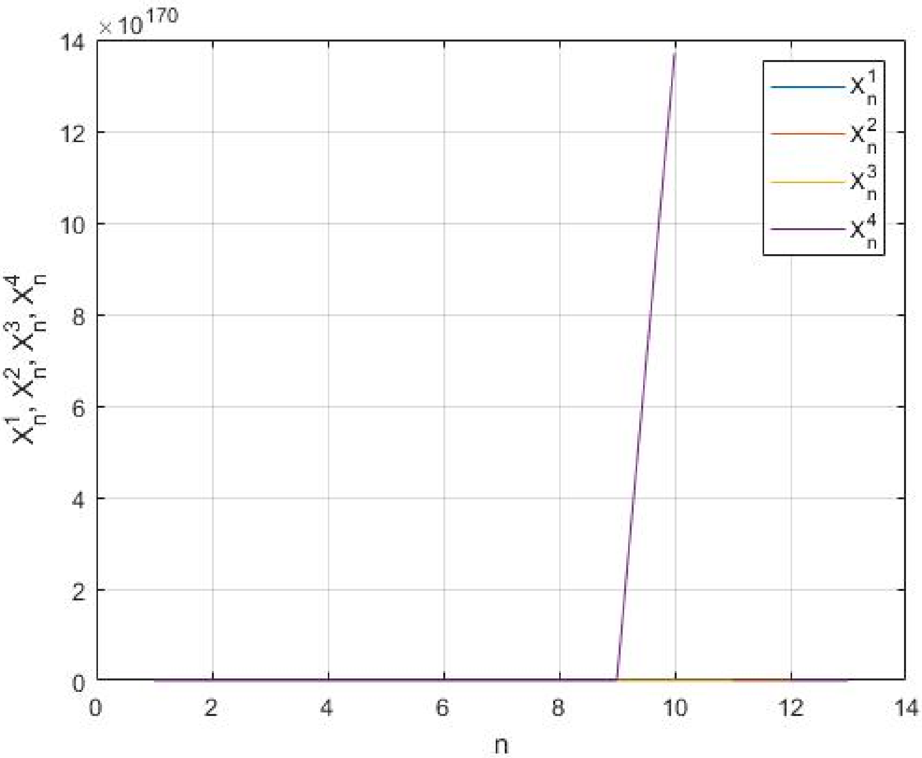

Figure 1.5: Plot of system (1.35) using the initial values (1.34).

Chapter 2

On a symmetric system of higher-order difference equations

2.1 Introduction

Difference equations and systems of difference equations are practically utilized across diverse fields such as engineering, biology, economics, medicine, computer science and more. Some particularly intriguing instances within this realm are symmetric systems and close-to-symmetric difference equations systems. This concept is exemplified by works like [1, 12, 15, 16, 19, 25, 27, 29, 32, 57, 59, 61, 62].

This chapter is based on our previous publication [5], in which we addressed a study about a symmetric difference equations system, analyzed the properties and examined the solutions' behavior of this system.

The problem presented in [20] is as follows:

Open problem. Does the given difference equation have a solution

$$x_{n+1} = \frac{\beta x_{n-1}}{\beta + x_n}, \quad x_{-1}, x_0 \geq 0, \quad \beta > 0, \quad n \in \mathbb{N}_0, \tag{2.1}$$

such that $\lim_{n\to\infty} x_n = 0$.

In [53], Stević provided a positive response to the open problem in the specific case where β equals 1. In this case, he examined the convergence, the periodicity, the monotonicity, and determined the limit of the solution for the equation presented below

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}, \quad x_{-1}, x_0 \geq 0, \quad n \in \mathbb{N}_0, \tag{2.2}$$

in specific cases and under special conditions.

In the same paper, the author presented another form of the solution formula, each term in the sequence was written in function of some previous terms. He combined the above-mentioned properties into a very important theorem, which is the same theorem that we are going to generalize in this chapter.

Moreover, in [53], Stević generalized the previous results to the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)}, \quad x_{-1}, x_0 \geq 0, \quad n \in \mathbb{N}_0, \tag{2.3}$$

where g is a function that satisfies these conditions

(a) $g \in C^1(\mathbb{R}_+)$,

(b) $g(0) = 1$,

(c) $g'(x) > 0$, for $x \in \mathbb{R}_+$,

with $g(x) > 1$ for all $x \in \mathbb{R}_+ \setminus \{0\}$, and equation (2.3) has only non-negative equilibrium point which is $\overline{x} = 0$.

Stević gave the solution to equation (2.3) in these two cases:

Case 1. $x_{-1} = x_0 = 0$,

in that case $x_n = 0$, for all n $\in \mathbb{N}_{-1}$.

Case 2. $(x_{-1} = 0 \text{ and } x_0 \neq 0)$ or $(x_{-1} \neq 0 \text{ and } x_0 = 0)$, in this case equation (2.3) has a 2−periodic solution $(x_{-1}, x_0, x_{-1}, x_0, x_{-1}, \ldots)$.

Now, if $x_{-1}, x_0 > 0$, the solution of equation (2.3) is positive, and here, Stević didn't give an explicit formula to the solution, but he just studied some properties of equation (2.3) in some theorems.

So, we can conclude that the author in [53] answered to the open problem posed in [20] only in these two cases:

Case 1. $\beta = 1$.

Case 2. At least one of the initial values is equal to zero.

Additionally, in [51] the author studied the following higher-order difference equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad n, k \in \mathbb{N}_0. \tag{2.4}$$

Inspired by the above-mentioned studies, we are going to extend equations (2.2) and (2.4) to the following symmetric system of higher-order difference equations

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + y_{n-k}}, \quad y_{n+1} = \frac{y_{n-(2k+1)}}{1 + x_{n-k}}, \quad n, k \in \mathbb{N}_0, \tag{2.5}$$

the initial values $x_{-(2k+1)}, x_{-2k}, \ldots, x_0, y_{-(2k+1)}, y_{-2k}, \ldots, y_0$ are non-negative real numbers.

2.2 An expansion of the principal theorem outlined in [52]

In this section, we are going to introduce a significant theorem that is going to aid us in presenting outcomes related to system (2.6).

$$x_{n+1} = \frac{x_{n-1}}{1 + y_n}, \quad y_{n+1} = \frac{y_{n-1}}{1 + x_n}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Theorem 2.2.1 *Let's consider system (2.6). Suppose that the initial values x_{-1}, x_0, y_{-1} and y_0 satisfy this condition*

$$\min\{x_{-1}, x_0, y_{-1}, y_0\} > 0, \tag{2.7}$$

so, for any solution $\{(x_n, y_n)\}_{n\geq -1}$ to system(2.6) that satisfies condition (2.7) the following assertions are valid.

(*a*) *The subsequences $\{(x_{2n}, y_{2n})\}_{n\geq 0}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\geq -1}$ decrease and there are non-negative constants a_1, a_2, b_1, b_2, such that*

$$\lim_{n\to\infty}(x_{2n}, y_{2n}) = (a_1, a_2) \ \ and \ \ \lim_{n\to\infty}(x_{2n+1}, y_{2n+1}) = (b_1, b_2). \tag{2.8}$$

(*b*) *If a_1, a_2, b_1 and b_2 represent the numbers specified in (2.8), so the sequence described as*

$$\begin{array}{lcl} (x_{2n-1}, y_{2n-1}) & = & (b_1, b_2) \\ (x_{2n}, y_{2n}) & = & (a_1, a_2) \end{array}, n \in \mathbb{N}_0$$

constitutes a two-periodic solution of system (2.6).

(*c*) *The following relation*

$$a_1 b_2 = 0 \ \ and \ \ a_2 b_1 = 0,$$

is valid.

(*d*) *If there is $n_0 \in \mathbb{N}_0$, such that*

$$x_n \geq y_{n+1} \geq x_{n+2}, \ and \ y_n \geq x_{n+1} \geq y_{n+2}, \ \ for \ \ n \geq n_0,$$

then

$$\lim_{n\to\infty}(x_n, y_n) = (0, 0).$$

(*e*) *The following formulas*

$$x_{2n} = x_0\left[1 - y_1 \sum_{j=1}^{n} \prod_{i=1}^{j-1} \frac{1}{1 + x_{2i}} \prod_{k=1}^{j} \frac{1}{1 + y_{2k-1}}\right], n \geq 0; \tag{2.9}$$

$$y_{2n} = y_0 \left[1 - x_1 \sum_{j=1}^{n} \prod_{i=1}^{j-1} \frac{1}{1+y_{2i}} \prod_{k=1}^{j} \frac{1}{1+x_{2k-1}} \right], n \geq 0; \tag{2.10}$$

$$x_{2n+1} = x_{-1} \left[1 - \frac{y_0}{1+y_0} \sum_{j=0}^{n} \prod_{i=1}^{j} \frac{1}{1+x_{2i-1}} \prod_{k=1}^{j} \frac{1}{1+y_{2k}} \right], n \geq -1; \tag{2.11}$$

$$y_{2n+1} = y_{-1} \left[1 - \frac{x_0}{1+x_0} \sum_{j=0}^{n} \prod_{i=1}^{j} \frac{1}{1+y_{2i-1}} \prod_{k=1}^{j} \frac{1}{1+x_{2k}} \right], n \geq -1. \tag{2.12}$$

are valid.

(f) *If*

$$x_0 + x_0^2 \leq y_{-1} \ \textit{and} \ y_0 + y_0^2 \leq x_{-1}, \tag{2.13}$$

then

$x_{2n} \to a_1 = 0$, $y_{2n} \to a_2 = 0$, $x_{2n+1} \to b_1 \neq 0$ *and* $y_{2n+1} \to b_2 \neq 0$,

as n *tends to the infinity.*

(g) *Suppose that a solution of the system* (2.6) *converges to zero, then there exists* m_0 *in the set of natural numbers* $\mathbb{N}_0$*, such that*

$$y_{n+2} < x_{n+1} \ \textit{and} \ x_{n+2} < y_{n+1}, \ \textit{for all} \ n \in \mathbb{N}_{m_0}.$$

Proof.

(a) From system (2.6), we have

$$x_{n+1} < x_{n-1} \ \text{and} \ y_{n+1} < y_{n-1}, \ \text{for} \ n \in \mathbb{N}_0,$$

so, $\{(x_{2n}, y_{2n})\}_{n\geq 0}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\geq -1}$ decrease.

Since the sequences decrease and comprise positive terms, they converge(a decreasing sequence that is lower bounded is convergent). Hence, there are a_1, a_2, b_1, $b_2 \geq 0$, such that

$$\lim_{n\to\infty}(x_{2n}, y_{2n}) = (a_1, a_2) \text{ and } \lim_{n\to\infty}(x_{2n+1}, y_{2n+1}) = (b_1, b_2).$$

$(b)-(c)$ Suppose that $\{(x_n, y_n)\}_{n\geq -1}$ represents a two-periodic solution to system (2.6). Thus, from (2.8) and system (2.6) we get

$$\left(a_1 = \frac{a_1}{1+b_2} \text{ and } a_2 = \frac{a_2}{1+b_1}\right) \text{or} \left(b_1 = \frac{b_1}{1+a_2} \text{ and } b_2 = \frac{b_2}{1+a_1}\right),$$

in other words

$$(a_1 + a_1 b_2 = a_1 \text{ and } a_2 + a_2 b_1 = a_2) \text{ or } (b_1 + b_1 a_2 = b_1 \text{ and } b_2 + b_2 a_1 = b_2),$$

and therefore, $a_1 b_2 = 0$ and $a_2 b_1 = 0$ are simultaneously checked.

(d) Suppose that there is an $n_0 \in \mathbb{N}_0$, such that

$$x_n \geq y_{n+1} \geq x_{n+2}, \text{ and } y_n \geq x_{n+1} \geq y_{n+2}, \text{ for all } n \geq n_0,$$

using (2.8) and by passing to the limit as n approaches infinity, we obtain

$$a_1 \geq b_2 \geq a_1 \geq b_2 \geq \ldots \geq 0, \text{ and } a_2 \geq b_1 \geq a_2 \geq b_1 \geq \ldots \geq 0, \tag{2.14}$$

or

$$b_1 \geq a_2 \geq b_1 \geq a_2 \geq \ldots \geq 0, \text{ and } b_2 \geq a_1 \geq b_2 \geq a_1 \geq \ldots \geq 0. \tag{2.15}$$

By combining (2.14), (2.15)with the outcome from theorem (2.2.1) (c), we establish that

$$a_1 = a_2 = b_1 = b_2 = 0,$$

which consequently implies

$$\lim_{n\to\infty}(x_n, y_n) = (0,0)\,.$$

It's worth noting that the outcome derived in the theorem (2.2.1) (c) is similar to

($a_1 = 0$ and $a_2 = 0$) or ($a_1 = 0$ and $b_1 = 0$) or ($a_2 = 0$ and $b_2 = 0$) or ($b_1 = 0$ and $b_2 = 0$).

(e) System (2.6) yields

$$\begin{aligned}
x_1 &= \frac{x_{-1}}{1+y_0} = x_{-1} - \frac{x_{-1}y_0}{1+y_0} = x_{-1}\left[1-\frac{y_0}{1+y_0}\right]\\
x_3 &= \frac{x_1}{1+y_2} = x_1 - \frac{x_1y_2}{1+y_2}\\
&= x_{-1}\left[1-\frac{y_0}{1+y_0}\right] - \frac{x_{-1}}{1+y_0}\frac{y_0}{1+x_1}\frac{1}{1+y_2}\\
x_3 &= x_{-1}\left[1-\frac{y_0}{1+y_0}\left(1+\frac{1}{1+x_1}\frac{1}{1+y_2}\right)\right]\\
x_5 &= \frac{x_3}{1+y_4} = x_3 - \frac{x_3y_4}{1+y_4}\\
&= x_{-1}\left[1-\frac{y_0}{1+y_0}\left(1+\frac{1}{1+x_1}\frac{1}{1+y_2}\right)\right] - \frac{x_1}{1+y_2}\frac{y_2}{1+x_3}\frac{1}{1+y_4}\\
&= x_{-1}\left[1-\frac{y_0}{1+y_0}\left(1+\frac{1}{1+x_1}\frac{1}{1+y_2}\right)\right] - \frac{x_{-1}}{1+y_0}\frac{1}{1+y_2}\frac{y_0}{1+x_1}\frac{1}{1+x_3}\frac{1}{1+y_4},
\end{aligned}$$

so

$$x_5 = x_{-1}\left[1-\frac{y_0}{1+y_0}\left(1+\frac{1}{1+x_1}\frac{1}{1+y_2}+\frac{1}{1+x_1}\frac{1}{1+y_2}\frac{1}{1+x_3}\frac{1}{1+y_4}\right)\right].$$

By induction, we can get

$$x_{2n+1} = x_{-1}\left[1-\frac{y_0}{1+y_0}\sum_{j=0}^{n}\prod_{i=1}^{j}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+y_{2k}}\right], n \geq -1.$$

Likewise, from system (2.6), we obtain

$$
\begin{aligned}
x_2 &= \frac{x_0}{1+y_1} = x_0 - \frac{x_0 y_1}{1+y_1} = x_0\left[1 - \frac{y_1}{1+y_1}\right] \\
x_4 &= \frac{x_2}{1+y_3} = x_2 - \frac{x_2 y_3}{1+y_3} \\
&= x_0\left[1 - \frac{y_1}{1+y_1}\right] - \frac{x_0}{1+y_1}\frac{y_1}{1+x_2}\frac{1}{1+y_3} \\
x_4 &= x_0\left[1 - y_1\left(\frac{1}{1+y_1} + \frac{1}{1+y_1}\frac{1}{1+x_2}\frac{1}{1+y_3}\right)\right] \\
x_6 &= \frac{x_4}{1+y_5} = x_4 - \frac{x_4 y_5}{1+y_5} \\
&= x_0\left[1 - y_1\left(\frac{1}{1+y_1} + \frac{1}{1+y_1}\frac{1}{1+x_2}\frac{1}{1+y_3}\right)\right] - \frac{x_2}{1+y_3}\frac{y_3}{1+x_4}\frac{1}{1+y_5} \\
&= x_0\left[1 - y_1\left(\frac{1}{1+y_1} + \frac{1}{1+y_1}\frac{1}{1+x_2}\frac{1}{1+y_3}\right)\right] - \frac{x_0}{1+y_1}\frac{1}{1+y_3}\frac{y_1}{1+x_2}\frac{1}{1+x_4}\frac{1}{1+y_5},
\end{aligned}
$$

so

$$
x_6 = x_0\left[1 - y_1\left(1 + \frac{1}{1+y_1} + \frac{1}{1+y_1}\frac{1}{1+x_2}\frac{1}{1+y_3} + \frac{1}{1+y_1}\frac{1}{1+x_2}\frac{1}{1+y_3}\frac{1}{1+x_4}\frac{1}{1+y_5}\right)\right].
$$

By induction, we can get

$$
x_{2n} = x_0\left[1 - y_1 \sum_{j=1}^{n}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}}\right], n \geq 0.
$$

Now, we are going to demonstrate the validity of relations (2.9) and (2.11).

- With a quick calculation, we confirm that relation (2.9) holds for $n = 0$.

- Assuming that it is verified at the order n, namely

$$
x_{2n} = x_0\left[1 - y_1 \sum_{j=1}^{n}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}}\right].
$$

- We are going to demonstrate its validity for the order $(n+1)$. We have

$$\begin{aligned}
x_{2n+2} &= \frac{x_{2n}}{1+y_{2n+1}} \\
&= x_{2n} - \frac{x_{2n}y_{2n+1}}{1+y_{2n+1}} \\
&= x_{2n} - \frac{x_0}{1+y_1}\frac{y_1}{1+x_2}\frac{1}{1+y_3}\cdots\frac{1}{1+x_{2n}}\frac{1}{1+y_{2n+1}} \\
&= x_{2n} - x_0y_1\prod_{i=1}^{n}\frac{1}{1+x_{2i}}\prod_{k=1}^{n+1}\frac{1}{1+y_{2k-1}} \\
&= x_0\left[1 - y_1\sum_{j=1}^{n}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}}\right] - x_0y_1\prod_{i=1}^{n}\frac{1}{1+x_{2i}}\prod_{k=1}^{n+1}\frac{1}{1+y_{2k-1}} \\
x_{2n+2} &= x_0\left[1 - y_1\left(\sum_{j=1}^{n}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}} + \prod_{i=1}^{n}\frac{1}{1+x_{2i}}\prod_{k=1}^{n+1}\frac{1}{1+y_{2k-1}}\right)\right].
\end{aligned}$$

Hence

$$x_{2n+2} = x_0\left[1 - y_1\sum_{j=1}^{n+1}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}}\right].$$

Therefore, relation (2.9) is verified at the order $(n+1)$, implying its validity for $n \geq 0$.

Similarly, we are going to demonstrate the truth of relation (2.11).

- With a quick calculation, we confirm that relation (2.11) holds for $n = -1$.

- Assuming that it is verified at the order n, namely

$$x_{2n+1} = x_{-1}\left[1 - \frac{y_0}{1+y_0}\sum_{j=0}^{n}\prod_{i=1}^{j}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+y_{2k}}\right].$$

- We are going to demonstrate its validity for the order $(n+1)$. We have

$$\begin{aligned}
x_{2n+3} &= \frac{x_{2n+1}}{1+y_{2n+2}} \\
&= x_{2n+1} - \frac{x_{2n+1}y_{2n+2}}{1+y_{2n+2}} \\
&= x_{2n+1} - \frac{x_{-1}}{1+y_0}\frac{y_0}{1+x_1}\frac{1}{1+y_2}\frac{1}{1+x_3}\cdots\frac{1}{1+x_{2n+1}}\frac{1}{1+y_{2n+2}} \\
&= x_{2n+1} - \frac{x_{-1}y_0}{1+y_0}\prod_{i=1}^{n+1}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{n+1}\frac{1}{1+y_{2k}} \\
&= x_{-1}\left[1 - \frac{y_0}{1+y_0}\sum_{j=0}^{n}\prod_{i=1}^{j}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+y_{2k}}\right] - \frac{x_{-1}y_0}{1+y_0}\prod_{i=1}^{n+1}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{n+1}\frac{1}{1+y_{2k}} \\
x_{2n+3} &= x_{-1}\left[1 - \frac{y_0}{1+y_0}\left(\sum_{j=0}^{n}\prod_{i=1}^{j}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+y_{2k}} + \prod_{i=1}^{n+1}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{n+1}\frac{1}{1+y_{2k}}\right)\right].
\end{aligned}$$

Hence

$$x_{2n+3} = x_{-1}\left[1 - \frac{y_0}{1+y_0}\sum_{j=0}^{n+1}\prod_{i=1}^{j}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+y_{2k}}\right].$$

Therefore, relation (2.11) is verified at the order $(n+1)$, implying its validity for $n \geq -1$.

The proofs for relations (2.10) and (2.12) are analogous to the previous one and will be skipped.

(f) Relation (2.13) can be rephrased as

$$x_0 \leq y_1 \text{ and } y_0 \leq x_1. \tag{2.16}$$

It's important to note that

- if $x_0 + x_0^2 \leq y_{-1}$, then either $(x_{2n})_{n\geq 0}$ or $(y_{2n+1})_{n\geq -1}$ has a non-zero limit.
- if $y_0 + y_0^2 \leq x_{-1}$, then either $(x_{2n+1})_{n\geq -1}$ or $(y_{2n})_{n\geq 0}$ has a non-zero limit.

Effectively, if we set $a_1 = b_2 = 0$, then from relations (2.9) and (2.12), we obtain

$$\frac{1}{y_1} = \sum_{j=1}^{\infty}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}},$$

since $x_{2k} > 0$ imply that $0 < \dfrac{1}{1+x_{2k}} < 1$, we can obtain

$$\prod_{k=1}^{j}\frac{1}{1+x_{2k}} = \frac{1}{1+x_{2j}}\prod_{k=1}^{j-1}\frac{1}{1+x_{2k}} < \prod_{k=1}^{j-1}\frac{1}{1+x_{2k}},$$

hence

$$\begin{aligned}\frac{1}{y_1} &= \sum_{j=1}^{\infty}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}}\\ &> \sum_{j=1}^{\infty}\prod_{k=1}^{j}\frac{1}{1+y_{2i-1}}\prod_{i=1}^{j}\frac{1}{1+x_{2k}}\\ &= \frac{1+x_0}{x_0} - 1 = \frac{1}{x_0},\end{aligned}$$

so, we get

$$\frac{1}{y_1} > \frac{1}{x_0},$$

therefore, $y_1 < x_0$ (contradiction with (2.16)).

If we put $a_2 = b_1 = 0$, we obtain from relations (2.10) and (2.11), that

$$\frac{1}{x_1} = \sum_{j=1}^{\infty}\prod_{i=1}^{j-1}\frac{1}{1+y_{2i}}\prod_{k=1}^{j}\frac{1}{1+x_{2k-1}},$$

since $y_{2k} > 0$ imply that $0 < \dfrac{1}{1+y_{2k}} < 1$, we can obtain

$$\prod_{k=1}^{j}\frac{1}{1+y_{2k}} = \frac{1}{1+y_{2j}}\prod_{k=1}^{j-1}\frac{1}{1+y_{2k}} < \prod_{k=1}^{j-1}\frac{1}{1+y_{2k}},$$

hence

$$\begin{aligned}\frac{1}{x_1} &= \sum_{j=1}^{\infty}\prod_{i=1}^{j-1}\frac{1}{1+y_{2i}}\prod_{k=1}^{j}\frac{1}{1+x_{2k-1}}\\ &> \sum_{j=1}^{\infty}\prod_{i=1}^{j}\frac{1}{1+x_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+y_{2k}}\\ &= \frac{1+y_0}{y_0}-1=\frac{1}{y_0},\end{aligned}$$

so, we get

$$\frac{1}{x_1} > \frac{1}{y_0},$$

thus, $x_1 < y_0$ (contradiction with (2.16)).

Now, using (2.16) and some calculations, we obtain

$$
\begin{aligned}
y_3 - x_2 &= \frac{y_1}{1+x_2} - x_2 \\
&= \frac{y_1 - x_2 - x_2^2}{1+x_2} \\
&= \frac{1}{1+x_2}\left[y_1 - x_2 - x_2^2\right] \\
&= \frac{1}{1+x_2}\left[y_1 - \frac{x_0}{1+y_1} - \left(\frac{x_0}{1+y_1}\right)^2\right] \\
&= \frac{1}{1+x_2}\left[\frac{y_1(1+y_1)^2 - x_0(1+y_1) - x_0^2}{(1+y_1)^2}\right] \\
&= \frac{1}{1+x_2}\left[\frac{y_1 + y_1^3 + 2y_1^2 - x_0 - x_0 y_1 - x_0^2}{(1+y_1)^2}\right] \\
&= \frac{1}{1+x_2}\left[\frac{y_1^3 + y_1^2 - x_0^2 + y_1(y_1 - x_0) + y_1 - x_0}{(1+y_1)^2}\right] \\
&\geq \frac{y_1^3}{(1+x_2)(1+y_1)^2} > 0,
\end{aligned}
$$

and (2.17)

$$
\begin{aligned}
x_3 - y_2 &= \frac{x_1}{1+y_2} - y_2 \\
&= \frac{x_1 - y_2 - y_2^2}{1+y_2} \\
&= \frac{1}{1+y_2}\left[x_1 - y_2 - y_2^2\right] \\
&= \frac{1}{1+y_2}\left[x_1 - \frac{y_0}{1+x_1} - \left(\frac{y_0}{1+x_1}\right)^2\right] \\
&= \frac{1}{1+y_2}\left[\frac{x_1(1+x_1)^2 - y_0(1+x_1) - y_0^2}{(1+x_1)^2}\right] \\
&= \frac{1}{1+y_2}\left[\frac{x_1 + x_1^3 + 2x_1^2 - y_0 - y_0 x_1 - y_0^2}{(1+x_1)^2}\right] \\
&= \frac{1}{1+y_2}\left[\frac{x_1^3 + x_1^2 - y_0^2 + x_1(x_1 - y_0) + x_1 - y_0}{(1+x_1)^2}\right] \\
&\geq \frac{x_1^3}{(1+y_2)(1+x_1)^2} > 0.
\end{aligned}
$$

Assuming that

$$y_{2n-1} > x_{2n-2} \text{ and } x_{2n-1} > y_{2n-2}, \tag{2.18}$$

then, we have

$$\begin{aligned}
y_{2n+1}-x_{2n} &= \frac{y_{2n-1}}{1+x_{2n}}-x_{2n}\\
&= \frac{y_{2n-1}-x_{2n}-x_{2n}^{2}}{1+x_{2n}}\\
&= \frac{1}{1+x_{2n}}\left[y_{2n-1}-x_{2n}-x_{2n}^{2}\right]\\
&= \frac{1}{1+x_{2n}}\left[y_{2n-1}-\frac{x_{2n-2}}{1+y_{2n-1}}-\left(\frac{x_{2n-2}}{1+y_{2n-1}}\right)^{2}\right]\\
&= \frac{1}{1+x_{2n}}\left[\frac{y_{2n-1}\left(1+y_{2n-1}\right)^{2}-x_{2n-2}\left(1+y_{2n-1}\right)-x_{2n-2}^{2}}{\left(1+y_{2n-1}\right)^{2}}\right]\\
&= \frac{1}{1+x_{2n}}\left[\frac{y_{2n-1}+y_{2n-1}^{3}+2y_{2n-1}^{2}-x_{2n-2}-x_{2n-2}y_{2n-1}-x_{2n-2}^{2}}{\left(1+y_{2n-1}\right)^{2}}\right]\\
&= \frac{1}{1+x_{2n}}\left[\frac{y_{2n-1}^{3}+y_{2n-1}^{2}-x_{2n-2}^{2}+y_{2n-1}(y_{2n-1}-x_{2n-2})+y_{2n-1}-x_{2n-2}}{\left(1+y_{2n-1}\right)^{2}}\right]\\
&\geq \frac{y_{2n-1}^{3}}{\left(1+x_{2n}\right)\left(1+y_{2n-1}\right)^{2}}>0,
\end{aligned}$$

and

$$\begin{aligned}
x_{2n+1}-y_{2n} &= \frac{x_{2n-1}}{1+y_{2n}}-y_{2n}\\
&= \frac{x_{2n-1}-y_{2n}-y_{2n}^{2}}{1+y_{2n}}\\
&= \frac{1}{1+y_{2n}}\left[x_{2n-1}-y_{2n}-y_{2n}^{2}\right]\\
&= \frac{1}{1+y_{2n}}\left[x_{2n-1}-\frac{y_{2n-2}}{1+x_{2n-1}}-\left(\frac{y_{2n-2}}{1+x_{2n-1}}\right)^{2}\right]\\
&= \frac{1}{1+y_{2n}}\left[\frac{x_{2n-1}\left(1+x_{2n-1}\right)^{2}-y_{2n-2}\left(1+x_{2n-1}\right)-y_{2n-2}^{2}}{\left(1+x_{2n-1}\right)^{2}}\right]\\
&= \frac{1}{1+y_{2n}}\left[\frac{x_{2n-1}+x_{2n-1}^{3}+2x_{2n-1}^{2}-y_{2n-2}-y_{2n-2}x_{2n-1}-y_{2n-2}^{2}}{\left(1+x_{2n-1}\right)^{2}}\right]\\
&= \frac{1}{1+y_{2n}}\left[\frac{x_{2n-1}^{3}+x_{2n-1}^{2}-y_{2n-2}^{2}+x_{2n-1}(x_{2n-1}-y_{2n-2})+x_{2n-1}-y_{2n-2}}{\left(1+x_{2n-1}\right)^{2}}\right]\\
&\geq \frac{x_{2n-1}^{3}}{\left(1+y_{2n}\right)\left(1+x_{2n-1}\right)^{2}}>0.
\end{aligned} \tag{2.19}$$

From (2.17), (2.18), (2.19) and by employing the method of induction, we get

$$y_{2n-1}>x_{2n-2} \ \text{ and } \ x_{2n-1}>y_{2n-2}, \quad \text{for all } \ n\geq 2.$$

From (2.18), we have

$$b_2 = \lim_{n\to\infty} y_{2n-1} \geq \lim_{n\to\infty} x_{2n-2} = a_1, \tag{2.20}$$

and

$$b_1 = \lim_{n\to\infty} x_{2n-1} \geq \lim_{n\to\infty} y_{2n-2} = a_2. \tag{2.21}$$

From (2.20), (2.21) and theorem (2.2.1) part (c), along with the initial note in the proof of (f), we get

$$\lim_{n\to\infty} x_{2n-2} = a_1 = 0, \quad \lim_{n\to\infty} y_{2n-2} = a_2 = 0,$$

and

$$\lim_{n\to\infty} x_{2n-1} = b_1 \neq 0, \quad \lim_{n\to\infty} y_{2n-1} = b_2 \neq 0.$$

(g) By shifting, we can see that for some $m_0 \in \mathbb{N}_0$

- If $x_{m_0+1} + x^2_{m_0+1} \leq y_{m_0}$ then $a_1 = 0$ or $b_1 = 0$.
- If $y_{m_0+1} + y^2_{m_0+1} \leq x_{m_0}$ then $a_2 = 0$ or $b_2 = 0$.

Therefore, if

$$\lim_{n\to\infty} (x_n, y_n) = (0, 0)\,,$$

then

$$x_{m_0+1} + x^2_{m_0+1} > y_{m_0} \text{ and } y_{m_0+1} + y^2_{m_0+1} > x_{m_0}, \quad m_0 \in \mathbb{N}_0.$$

Thus, for each $n \in \mathbb{N}_{m_0}$, we get

$$y_n < x_{n+1} + x^2_{n+1} \text{ and } x_n < y_{n+1} + y^2_{n+1},$$

this is similar to

$$y_{n+2} = \frac{y_n}{1 + x_{n+1}} < x_{n+1} \text{ and } x_{n+2} = \frac{x_n}{1 + y_{n+1}} < y_{n+1},$$

for all $n \in \mathbb{N}_{m_0}$.

The theorem's proof is now concluded.

■

Now, assuming condition (2.13) is satisfied, then by utilizing the fact that $\lim_{n\to\infty} x_{2n} = 0$, and letting n tends to infinity in (2.9), we obtain

$$\sum_{j=1}^{\infty}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}} = \frac{1}{y_1}. \tag{2.22}$$

We can represent relation (2.12) as follows

$$\sum_{j=1}^{n}\prod_{i=1}^{j}\frac{1}{1+y_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+x_{2k}} = \left(\frac{1+x_0}{x_0}\right)\left(1-\frac{y_{2n+1}}{y_{-1}}\right)-1. \tag{2.23}$$

Then

$$\sum_{j=1}^{n}\prod_{i=1}^{j}\frac{1}{1+y_{2i-1}}\prod_{k=1}^{j}\frac{1}{1+x_{2k}} < \sum_{j=1}^{n}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}} < \sum_{j=1}^{\infty}\prod_{i=1}^{j-1}\frac{1}{1+x_{2i}}\prod_{k=1}^{j}\frac{1}{1+y_{2k-1}}, \tag{2.24}$$

for every n$\in \mathbb{N}_0$.

From (2.22), (2.23) and (2.24), we obtain

$$0 < \left(\frac{1+x_0}{x_0}\right)\left(1-\frac{y_{2n+1}}{y_{-1}}\right)-1 < \frac{1}{y_1},$$

so

$$0 < 1-\frac{y_{2n+1}}{y_{-1}}-\frac{x_0}{1+x_0} < \frac{1+x_0}{y_{-1}}\frac{x_0}{1+x_0},$$

then

$$0 < y_{-1}-y_{2n+1}-\frac{y_{-1}x_0}{1+x_0} < x_0,$$

from where

$$0 < \frac{y_{-1}(1+x_0)-y_{-1}x_0}{1+x_0}-y_{2n+1} < x_0,$$

for n$\in \mathbb{N}_0$, which gives us

$$0 < \frac{y_{-1}}{1+x_0}-y_{2n+1} < x_0.$$

Based on the preceding outcomes and (2.16), we get

$$0 \leq y_1 - x_0 < y_{2n+1}. \tag{2.25}$$

Similarly, if condition (2.13) is satisfied, then by utilizing the fact that $\lim_{n\to\infty} y_{2n} = 0$ and letting n tends to infinity in (2.10), we obtain

$$\sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \frac{1}{1+y_{2i}} \prod_{k=1}^{j} \frac{1}{1+x_{2k-1}} = \frac{1}{x_1}. \tag{2.26}$$

We can represent relation (2.11) as follows

$$\sum_{j=1}^{n} \prod_{i=1}^{j} \frac{1}{1+x_{2i-1}} \prod_{k=1}^{j} \frac{1}{1+y_{2k}} = \left(\frac{1+y_0}{y_0}\right)\left(1 - \frac{x_{2n+1}}{x_{-1}}\right) - 1. \tag{2.27}$$

Thus

$$\sum_{j=1}^{n} \prod_{i=1}^{j} \frac{1}{1+x_{2i-1}} \prod_{k=1}^{j} \frac{1}{1+y_{2k}} < \sum_{j=1}^{n} \prod_{i=1}^{j-1} \frac{1}{1+y_{2i}} \prod_{k=1}^{j} \frac{1}{1+x_{2k-1}} < \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \frac{1}{1+y_{2i}} \prod_{k=1}^{j} \frac{1}{1+x_{2k-1}}, \tag{2.28}$$

for every n$\in \mathbb{N}_0$.

From (2.26), (2.27) and (2.28), we obtain

$$0 < \left(\frac{1+y_0}{y_0}\right)\left(1 - \frac{x_{2n+1}}{x_{-1}}\right) - 1 < \frac{1}{x_1},$$

so

$$0 < 1 - \frac{x_{2n+1}}{x_{-1}} - \frac{y_0}{1+y_0} < \frac{1+y_0}{x_{-1}} \frac{y_0}{1+y_0},$$

then

$$0 < x_{-1} - x_{2n+1} - \frac{x_{-1}y_0}{1+y_0} < y_0,$$

from where

$$0 < \frac{x_{-1}(1+y_0) - x_{-1}y_0}{1+y_0} - x_{2n+1} < y_0,$$

for n$\in \mathbb{N}_0$, which gives us

$$0 < \frac{x_{-1}}{1+y_0} - x_{2n+1} < y_0.$$

Building on the earlier findings and (2.16), we get

$$0 \leq x_1 - y_0 < x_{2n+1}. \tag{2.29}$$

Proposition 2.2.1 *Consider $\{(x_n, y_n)\}_{n\geq -1}$ as a solution to system (2.6). Let's suppose that the values x_0, y_0, x_1 and y_1 satisfy these conditions*

$$x_1 - y_0 \geq 0 \ \ \textit{and} \ \ y_1 - x_0 \geq 0.$$

Then

$$\lim_{n\to\infty} x_{2n+1} \neq x_1 - y_0 \ \textit{and} \ \lim_{n\to\infty} y_{2n+1} \neq y_1 - x_0.$$

Proof. There are two cases that need to be considered.

Case 1. When the equalities in (2.13) are satisfied, we get

$$x_1 - y_0 = 0 \ \ \text{and} \ \ y_1 - x_0 = 0.$$

Therefore, by utilizing (2.25), (2.29), along with the result derived in theorem (2.2.1) part (f), we obtain

$$\lim_{n\to\infty} x_{2n+1} = b_1 > x_1 - y_0 = 0 \text{ and } \lim_{n\to\infty} y_{2n+1} = b_2 > y_1 - x_0 = 0. \tag{2.30}$$

From where, if we assume the equalities in (2.13) are satisfied, then we get

$$\lim_{n\to\infty} x_{2n+1} = b_1 \neq x_1 - y_0 = 0 \text{ and } \lim_{n\to\infty} y_{2n+1} = b_2 \neq y_1 - x_0 = 0. \tag{2.31}$$

Case 2. When the strict inequalities in (2.13) are satisfied, we get

$$y_1 - x_0 > 0 \text{ and } x_1 - y_0 > 0.$$

Therefore, by utilizing relations (2.25), (2.29), along with the monotonicity of $\{(x_{2n+1})\}_{n\geq -1}$ and $\{(y_{2n+1})\}_{n\geq -1}$, we obtain

$$\lim_{n\to\infty} x_{2n+1} = b_1 \geq x_1 - y_0 > 0 \text{ and } \lim_{n\to\infty} y_{2n+1} = b_2 \geq y_1 - x_0 > 0. \tag{2.32}$$

Now, let's suppose, for instance, that

$$x_{-1} = 6, x_0 = 1, y_{-1} = 4 \text{ and } y_0 = 1, \tag{2.33}$$

within system (2.6). So we obtain the graph in Fig (2.1)

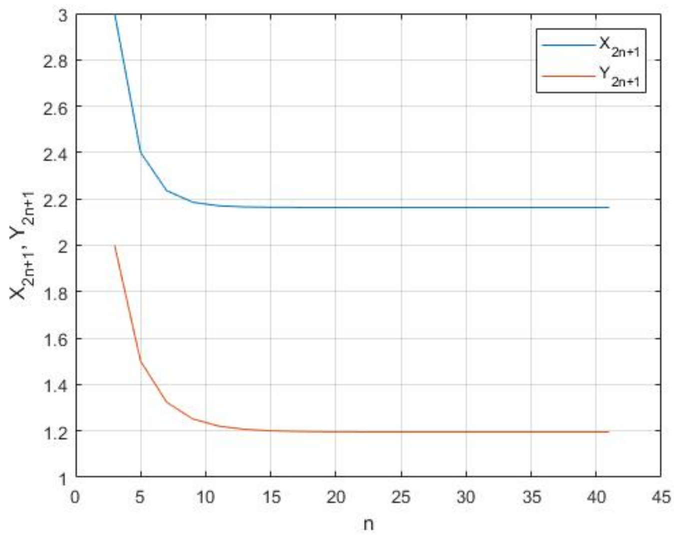

Figure 2.1: Plot of system (2.6) using the initial values (2.33).

From (2.33) and system (2.6), we can see that

$$x_1 - y_0 = 3 - 1 = 2 > 0 \text{ and } y_1 - x_0 = 2 - 1 = 1 > 0,$$

so we are in the second case.

Now, from the graph in Fig (2.1), it is easy to see that

$$\lim_{n\to\infty} x_{2n+1} \neq 2 = x_1 - y_0 \text{ and } \lim_{n\to\infty} y_{2n+1} \neq 1 = y_1 - x_0. \tag{2.34}$$

From where, When the strict inequalities in (2.13) are satisfied, we get

$$\lim_{n\to\infty} x_{2n+1} = b_1 \neq x_1 - y_0 > 0 \text{ and } \lim_{n\to\infty} y_{2n+1} = b_2 \neq y_1 - x_0 > 0. \tag{2.35}$$

Using (2.31) and (2.35), we obtain

$$\lim_{n\to\infty} x_{2n+1} \neq x_1 - y_0 \text{ and } \lim_{n\to\infty} y_{2n+1} \neq y_1 - x_0,$$

under the following condition

$$x_1 - y_0 \geq 0 \text{ and } y_1 - x_0 \geq 0.$$

■

2.3 Understanding system (2.5)

This section outlines the approach employed to streamline the analysis of the difference equations system (2.5), that is given by

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + y_{n-k}}, \quad y_{n+1} = \frac{y_{n-(2k+1)}}{1 + x_{n-k}}, \quad n, k \in \mathbb{N}_0.$$

Before we begin, it is important to mention that these difference equations have been studied in existing literature.

In [49, 50] Şimşek et al. studied the following equations

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.36}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{2.37}$$

$$x_{n+1} = \frac{x_{n-7}}{1 + x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{2.38}$$

Şimşek et al. examined the following generalization of equations (2.36)-(2.38) in [51]

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad n, k \in \mathbb{N}_0.$$

Motivated by the above mentioned works, we are going to introduce system(2.5) and study it.

From system (2.5), we can observe that x_{n+1} can be simply represented using $x_{n-(2k+1)}$ and y_{n-k},

(similarly: y_{n+1} is represented using $y_{n-(2k+1)}$ and x_{n-k}) .

Another observation is that the relation below holds

$$n + 1 - (n - k) = n - k - (n - (2k + 1)) = k + 1.$$

In other words, the difference between the indices of x_{n+1} and y_{n-k} on one hand,

and the indices of y_{n-k} and $x_{n-(2k+1)}$ on the other one,

(similarly: the difference between the indices of y_{n+1} and x_{n-k} on one hand,

and the indices of x_{n-k} and $y_{n-(2k+1)}$ on the other one) is equal to $k + 1$.

As a consequence, we can partition the set of indices into $k + 1$ distinct subsets, each subset being defined by

$$S_j = \{n \in \mathbb{N}_{-(2k+1)}, \ \ n = (k+1)m + j, \ \ m \geq -2\}, \ \ j = \overline{1, k+1}.$$

Thus, we can represent system (2.5) as follows

$$x_{(k+1)m+j} = \frac{x_{(k+1)(m-2)+j}}{1 + y_{(k+1)(m-1)+j}}, \quad y_{(k+1)m+j} = \frac{y_{(k+1)(m-2)+j}}{1 + x_{(k+1)(m-1)+j}}, \quad m \in \mathbb{N}_0, \tag{2.39}$$

for all $j = \overline{1, k+1}$.

Let's put

$$u_m^{(j)} = x_{(k+1)m+j}, \ \ v_m^{(j)} = y_{(k+1)m+j}, \ \ m \geq -2, \ \ j = \overline{1, k+1}, \tag{2.40}$$

so, from system (2.39) and relation (2.40), we obtain

$$u_m^{(j)} = \frac{u_{m-2}^{(j)}}{1 + v_{m-1}^{(j)}}, \ \ v_m^{(j)} = \frac{v_{m-2}^{(j)}}{1 + u_{m-1}^{(j)}}, \ \ m \geq 0, \ \ j = \overline{1, k+1},$$

from where, the sequences $\left\{\left(u_m^{(j)}, v_m^{(j)}\right)\right\}_{m \geq -2}$, $j = \overline{1, k+1}$ are $k+1$ solutions to this system

$$x_m = \frac{x_{m-2}}{1 + y_{m-1}}, \ \ y_m = \frac{y_{m-2}}{1 + x_{m-1}}, \ \ m \in \mathbb{N}_0. \tag{2.41}$$

Studying system (2.41) is similar to studying the system

$$x_{n+1} = \frac{x_{n-1}}{1 + y_n}, \ \ y_{n+1} = \frac{y_{n-1}}{1 + x_n}, \ \ n \in \mathbb{N}_0,$$

This system is essentially a simplified version of system (2.5) with $k = 0$.

The method employed demonstrates that the systems derived from system (2.5) with $k = 4, 5, 6, 7, 8, 9, 10$, respectively, are classified within the same problem category

$$x_{n+1} = \frac{x_{n-9}}{1 + y_{n-4}}, \ \ y_{n+1} = \frac{y_{n-9}}{1 + x_{n-4}}, \ \ n \in \mathbb{N}_0, \tag{2.42}$$

$$x_{n+1} = \frac{x_{n-11}}{1 + y_{n-5}}, \ \ y_{n+1} = \frac{y_{n-11}}{1 + x_{n-5}}, \ \ n \in \mathbb{N}_0, \tag{2.43}$$

$$x_{n+1} = \frac{x_{n-13}}{1 + y_{n-6}}, \ \ y_{n+1} = \frac{y_{n-13}}{1 + x_{n-6}}, \ \ n \in \mathbb{N}_0, \tag{2.44}$$

$$x_{n+1} = \frac{x_{n-15}}{1 + y_{n-7}}, \ \ y_{n+1} = \frac{y_{n-15}}{1 + x_{n-7}}, \ \ n \in \mathbb{N}_0, \tag{2.45}$$

$$x_{n+1} = \frac{x_{n-17}}{1 + y_{n-8}}, \ \ y_{n+1} = \frac{y_{n-17}}{1 + x_{n-8}}, \ \ n \in \mathbb{N}_0, \tag{2.46}$$

$$x_{n+1} = \frac{x_{n-19}}{1 + y_{n-9}}, \ \ y_{n+1} = \frac{y_{n-19}}{1 + x_{n-9}}, \ \ n \in \mathbb{N}_0, \tag{2.47}$$

$$x_{n+1} = \frac{x_{n-21}}{1 + y_{n-10}}, \quad y_{n+1} = \frac{y_{n-21}}{1 + x_{n-10}}, \quad n \in \mathbb{N}_0. \tag{2.48}$$

2.4 Numerical examples

Throughout this section, we are going to look at different concrete examples to better understand our theoretical outcomes. In particular, the examples cover various solutions' types that can emerge within the general system (2.6), such as periodic patterns and convergence. MATLAB is used to generate the plots in this section.

Example 2.4.1 *Let's examine system (2.6) with the following initial values*

$$x_{-1} = 2, \quad x_0 = 5, \quad y_{-1} = 4, \quad y_0 = 3. \tag{2.49}$$

This gives us the graph in Fig (2.2).

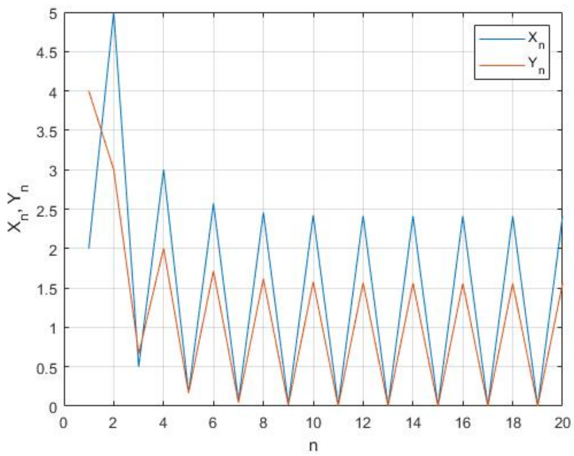

Figure 2.2: Plot of system (2.6) using the initial values (2.49).

Example 2.4.2 *Let's examine system (2.6) with the following initial values*

$$x_{-1} = 15, \quad x_0 = 10, \quad y_{-1} = 50, \quad y_0 = 15. \tag{2.50}$$

This gives us the graphs in Fig (2.3). These plots illustrate the monotonic behavior of $\{(x_{2n}, y_{2n})\}_{n\geq 0}$ *and* $\{(x_{2n+1}, y_{2n+1})\}_{n\geq -1}$*, as per the findings of Theorem (2.2.1)(b).*

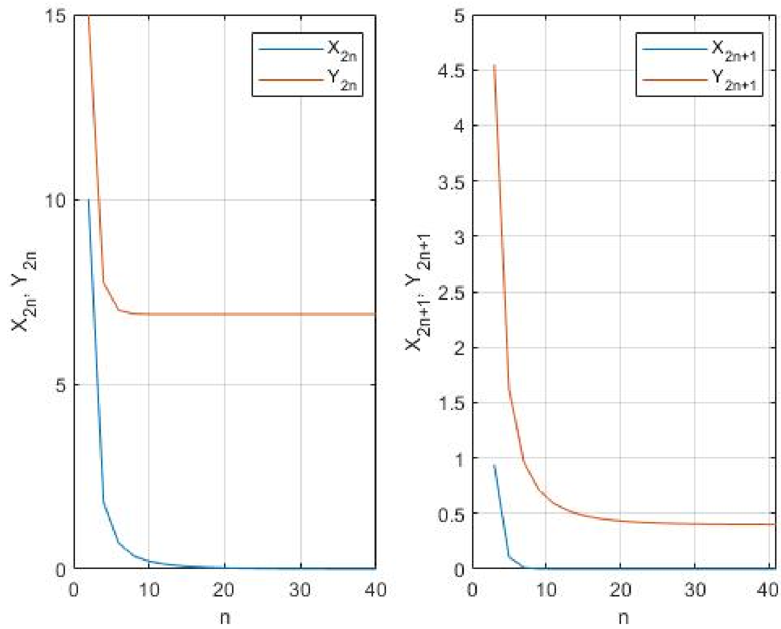

Figure 2.3: Plot of $\{(x_{2n}, y_{2n})\}_{n\geq 0}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\geq -1}$ using the initial values (2.50).

Example 2.4.3 *Let's examine system* (2.6) *with the following initial values*

$$x_{-1}^{(1)} = 2,\ \ x_0^{(1)} = 0,\ \ y_{-1}^{(1)} = 4,\ \ y_0^{(1)} = 0, \tag{2.51}$$

$$x_{-1}^{(2)} = 0,\ \ x_0^{(2)} = 0,\ \ y_{-1}^{(2)} = 5,\ \ y_0^{(2)} = 3, \tag{2.52}$$

$$x_{-1}^{(3)} = 2,\ \ x_0^{(3)} = 5,\ \ y_{-1}^{(3)} = 0,\ \ y_0^{(3)} = 0, \tag{2.53}$$

$$x_{-1}^{(4)} = 0,\ \ x_0^{(4)} = 5,\ \ y_{-1}^{(4)} = 0\ \ and\ \ y_0^{(4)} = 3. \tag{2.54}$$

This gives us the graphs in Fig (2.4)*. These plots illustrate the periodic nature of the solution for system* (2.6) *with the initial values* (2.51), (2.52), (2.53) *and* (2.54) *respectively, as per the findings of Theorem* (2.2.1)*(c).*

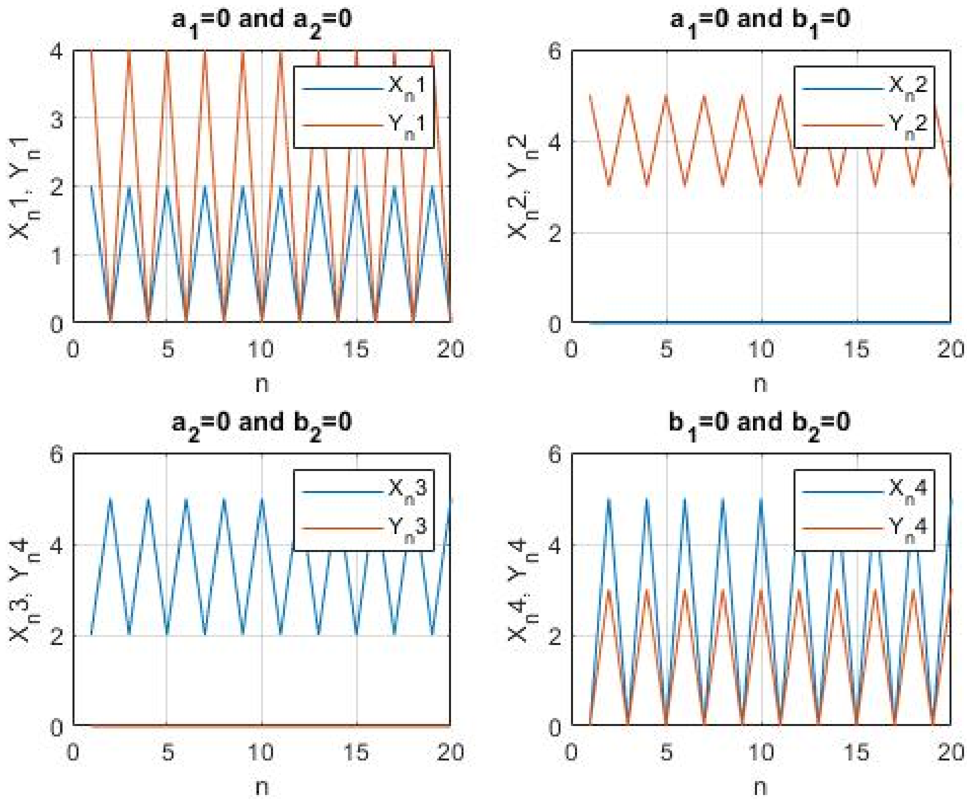

Figure 2.4: Plots of system (2.6) using the initial values (2.51), (2.52), (2.53) and (2.54) respectively.

Example 2.4.4 *Let's examine system* (2.6) *with the following initial values*

$$x_{-1} = 7, \;\; x_0 = 1, \;\; y_{-1} = 3, \;\; y_0 = 2. \tag{2.55}$$

This gives us the graphs in Fig (2.5). *These plots show the limits of* $\{(x_{2n}, y_{2n})\}_{n\geq 0}$ *and* $\{(x_{2n+1}, y_{2n+1})\}_{n\geq -1}$ *under the condition* (2.13)*, as per the findings of Theorem (1)(f).*

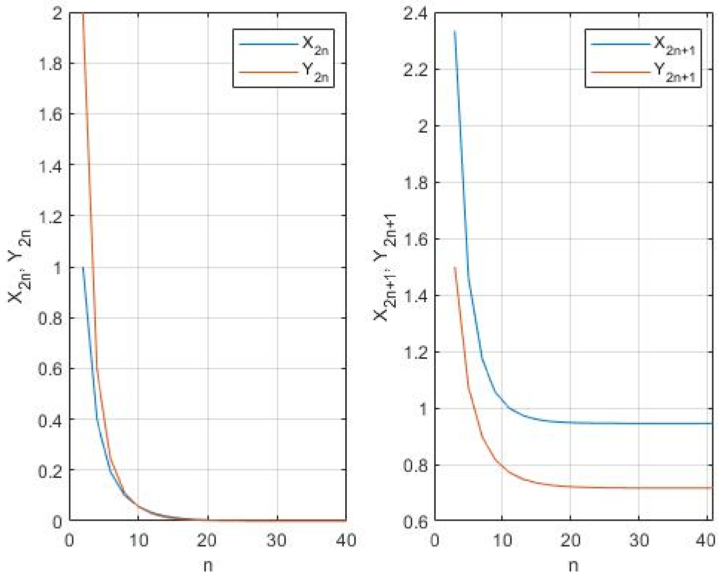

Figure 2.5: Plot of $\{(x_{2n}, y_{2n})\}_{n\geq 0}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\geq -1}$ using the initial values (2.55).

Chapter 3

Dynamical behavior of a possible discrete community model

3.1 Introduction

In recent years, numerous biological subjects have been represented through the use of difference equations. This approach has subsequently facilitated the examination of population dynamics and the influence of biotic factors on a significant scale.

Biotic factors encompass all the actions that living organisms directly exert on each other. These interactions are termed coactions and can be categorized into two distinct types:

- Homotypic (or intraspecific), when they occur between individuals of the same species.
- Heterotypic (or interspecific), when they occur between individuals of different species.

Heterotypic coactions type changes according to the scheme (3.1).

The Lotka-Volterra models in discrete-time, formulated by difference equations,

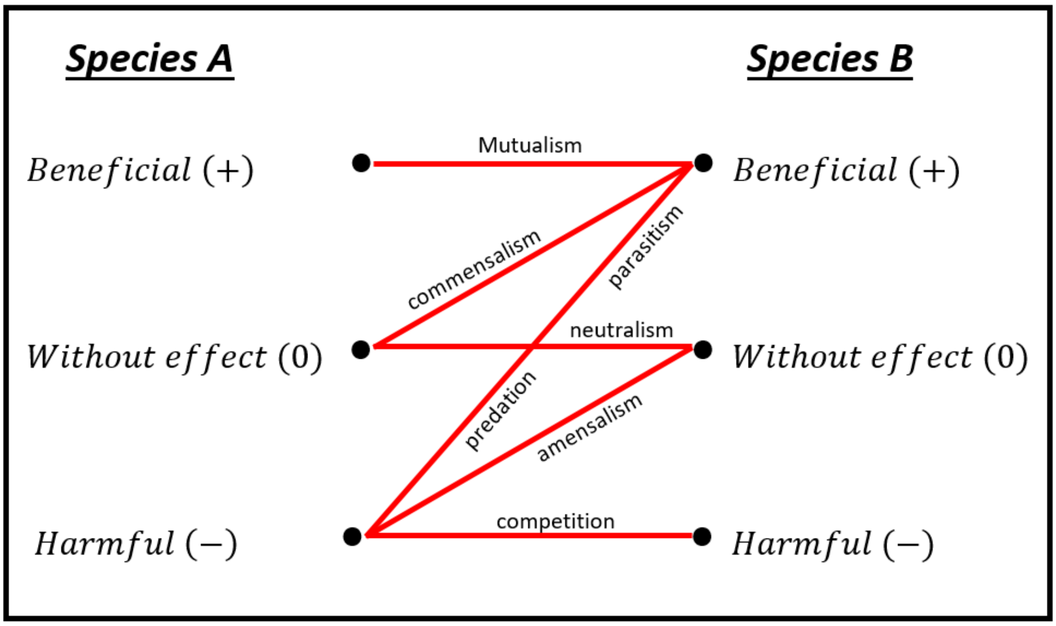

Figure 3.1: Heterotypic coactions types.

stand as one of the most celebrated models for population dynamics that study predation, which is one of the heterotypic coactions types (see [3, 8, 9, 13, 42, 44, 54, 63]).

One of the most interesting Lotka-Volterra predator-prey models is presented in [48] with an important study of the the solution's qualitative behavior to the following difference equations system

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \quad y_{n+1} = \frac{\delta y_n + \epsilon x_n y_n}{1 + \eta y_n}, \qquad n \in \mathbb{N}_0, \tag{3.1}$$

the parameters $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ and the initial values x_0 and y_0 are positive real numbers.

We distinguish between negative interactions that are harmful to the growth of individuals of the first species and positive interactions that promote the growth of individuals of the second species.

The signs $+$ and $-$ in system (3.1) clarify if the growth is favorable or unfavorable.

In this chapter, we are going to revisit and expand upon our research previously published in [7], titled 'Dynamical behavior of a possible discrete community model'.

So, we are going to generalize system (3.1) to the following community model

$$x_{n+1} = \frac{a_1 x_n - a_2 x_n y_n}{1 + a_3 x_n}, \quad y_{n+1} = \frac{a_4 y_n + a_5 y_n z_n}{1 + a_6 y_n}, \quad z_{n+1} = \frac{a_7 z_n + a_8 z_n x_n}{1 + a_9 z_n}, \qquad n \in \mathbb{N}_0, \tag{3.2}$$

the parameters a_i, $i = \overline{1,9}$ and the initial values x_0, y_0 and z_0 are positive real numbers.

System (3.2) presents interactions between individuals of three different species. Individuals of the second species inhibit the development of individuals of the first species, this interaction is called amensalism. Individuals of the second species benefit from individuals of the third species without harming them. Similarly, individuals of the third species benefit from individuals of the first species without harming them, this interaction is called commensalism.

Remark 3.1.1 *If we put $a_7 = a_4$, $a_8 = a_5$, $a_9 = a_6$ and $z_0 = y_0$, system (3.2) reduces to system (3.1).*

3.2 Justifying the choice of positive initial conditions

Consider system (3.2). Suppose that the parameters a_i, $i = \overline{1,9}$ are positive and the initial values x_0, y_0 and z_0 are non-negative real numbers.

Note that

- If $x_0 = 0$, so $x_n = 0$ for all $n \in \mathbb{N}_0$.
- If $y_0 = 0$, so $y_n = 0$ for all $n \in \mathbb{N}_0$.
- If $z_0 = 0$, so $z_n = 0$ for all $n \in \mathbb{N}_0$.

We distinguish the following cases

1. If $x_0 = y_0 = z_0 = 0$, so system (3.2) reduces to

$$x_n = y_n = z_n = 0, \text{ for all } n \in \mathbb{N}_0. \tag{3.3}$$

2. If $x_0 = y_0 = 0$, $z_0 \in]0, +\infty[$, so system (3.2) reduces to

$$z_{n+1} = \frac{a_7 z_n}{1 + a_9 z_n}, \quad n \in \mathbb{N}_0, \tag{3.4}$$

which is a Riccati equation, its solution and behavior are well-known.

3. If $x_0 = z_0 = 0$, $y_0 \in]0, +\infty[$, so system (3.2) reduces to

$$y_{n+1} = \frac{a_4 y_n}{1 + a_6 y_n}, \quad n \in \mathbb{N}_0, \tag{3.5}$$

which is a Riccati equation.

4. If $y_0 = z_0 = 0$, $x_0 \in]0, +\infty[$, so system (3.2) reduces to

$$x_{n+1} = \frac{a_1 x_n}{1 + a_3 x_n}, \quad n \in \mathbb{N}_0, \tag{3.6}$$

which is a Riccati equation.

5. If $x_0 = 0$, $y_0, z_0 \in]0, +\infty[$, so $x_n = 0$ for all $n \in \mathbb{N}_0$ and system (3.2) reduces to

$$y_{n+1} = \frac{a_4 y_n + a_5 y_n z_n}{1 + a_6 y_n}, \quad z_{n+1} = \frac{a_7 z_n}{1 + a_9 z_n}, \quad n \in \mathbb{N}_0. \tag{3.7}$$

As $y_0, z_0 > 0$, it follows that $y_n, z_n > 0$, for all $n \in \mathbb{N}_0$.

System (3.7) has only one equilibrium point $(\bar{y}, \bar{z})$ in $(]0, +\infty[)^2$, such that

$$\begin{aligned} \bar{y} &= \frac{a_5 (a_7 - 1) + a_9 (a_4 - 1)}{a_6 a_9}, \quad a_5 (a_7 - 1) + a_9 (a_4 - 1) > 0, \\ \bar{z} &= \frac{a_7 - 1}{a_9}, \quad a_7 > 1. \end{aligned}$$

Consider these two continuously differentiable functions

$$\begin{aligned} f_1 :]0, +\infty[\times]0, +\infty[&\to]0, +\infty[\\ (y, z) &\mapsto f_1(y, z) = \frac{a_4 y + a_5 y z}{1 + a_6 y}, \end{aligned}$$

$$\begin{aligned} f_2 :]0, +\infty[\times]0, +\infty[&\to]0, +\infty[\\ (y, z) &\mapsto f_2(y, z) = \frac{a_7 z}{1 + a_9 z}. \end{aligned}$$

So, the Jacobian matrix of the linearized system of (3.7) around $(\bar{y}, \bar{z})$ is given by

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial y}(\bar{y}, \bar{z}) & \dfrac{\partial f_1}{\partial z}(\bar{y}, \bar{z}) \\ \dfrac{\partial f_2}{\partial y}(\bar{y}, \bar{z}) & \dfrac{\partial f_2}{\partial z}(\bar{y}, \bar{z}) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{a_9}{a_4 a_9 + a_5(a_7 - 1)} & \dfrac{a_5(a_9(a_4 - 1) + a_5(a_7 - 1))}{a_6(a_4 a_9 + a_5(a_7 - 1))} \\ 0 & \dfrac{1}{a_7} \end{pmatrix}.$$

Additionally, the eigenvalues of the Jacobian matrix around $(\bar{y}, \bar{z})$ are given by
$\lambda_1 = \dfrac{a_9}{a_4 a_9 + a_5 (a_7 - 1)} < 1$,

$\lambda_2 = \dfrac{1}{a_7} < 1$,

hence, $(\bar{y}, \bar{z}) = \left(\dfrac{a_5 (a_7 - 1) + a_9 (a_4 - 1)}{a_6 a_9}, \dfrac{a_7 - 1}{a_9} \right)$ is locally asymptotically stable.

6. If $y_0 = 0$, $x_0, z_0 \in]0, +\infty[$, so $y_n = 0$ for all $n \in \mathbb{N}_0$ and system (3.2) reduces to

$$x_{n+1} = \frac{a_1 x_n}{1 + a_3 x_n}, \quad z_{n+1} = \frac{a_7 z_n + a_8 z_n x_n}{1 + a_9 z n}, \quad n \in \mathbb{N}_0. \tag{3.8}$$

As $x_0, z_0 > 0$, it follows that $x_n, z_n > 0$, for all $n \in \mathbb{N}_0$.

System (3.8) has only one equilibrium point $(\bar{x}, \bar{z})$ in $(]0, +\infty[)^2$, such that

$$\begin{aligned} \bar{x} &= \frac{a_1 - 1}{a_3}, \quad a_1 > 1, \\ \bar{z} &= \frac{a_3 (a_7 - 1) + a_8 (a_1 - 1)}{a_3 a_9}, \quad a_3 (a_7 - 1) + a_8 (a_1 - 1) > 0. \end{aligned}$$

Consider these two continuously differentiable functions

$g_1 :]0, +\infty[\times]0, +\infty[\to]0, +\infty[$

$$(x, z) \mapsto g_1 (x, z) = \frac{a_1 x}{1 + a_3 x},$$

$g_2 :]0, +\infty[\times]0, +\infty[\to]0, +\infty[$

$$(x, z) \mapsto g_2 (x, z) = \frac{a_7 z + a_8 z x}{1 + a_9 z}.$$

So, the Jacobian matrix of the linearized system of (3.8) around $(\bar{x}, \bar{z})$ is given by

$$\begin{pmatrix} \dfrac{\partial g_1}{\partial x}(\bar{x}, \bar{z}) & \dfrac{\partial g_1}{\partial z}(\bar{x}, \bar{z}) \\ \dfrac{\partial g_2}{\partial x}(\bar{x}, \bar{z}) & \dfrac{\partial g_2}{\partial z}(\bar{x}, \bar{z}) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{a_1} & 0 \\ \dfrac{a_8(a_3(a_7-1)+a_8(a_1-1))}{a_9(a_3a_7+a_8(a_1-1))} & \dfrac{a_3}{a_3a_7+a_8(a_1-1)} \end{pmatrix}.$$

Additionally, the eigenvalues of the Jacobian matrix around $(\bar{x}, \bar{z})$ are given by

$\lambda_1 = \dfrac{1}{a_1} < 1,$

$\lambda_2 = \dfrac{a_3}{a_3a_7 + a_8(a_1-1)} < 1,$

hence, $(\bar{x}, \bar{z}) = \left(\dfrac{a_1-1}{a_3}, \dfrac{a_3(a_7-1)+a_8(a_1-1)}{a_3a_9}\right)$ is locally asymptotically stable.

7. If $z_0 = 0$, $x_0, y_0 \in]0, +\infty[$, so $z_n = 0$ for all $n \in \mathbb{N}_0$ and system (3.2) reduces to

$$x_{n+1} = \frac{a_1x_n - a_2x_ny_n}{1+a_3x_n}, \quad y_{n+1} = \frac{a_4y_n}{1+a_6y_n}, \quad n \in \mathbb{N}_0, \tag{3.9}$$

Here, additional conditions must be imposed.

We have

$$x_1 = \frac{(a_1 - a_2y_0)x_0}{1+a_3x_0},$$

If $a_1 - a_2y_0 = 0$, so $x_1 = 0$ which imply $x_n = 0$ for all $n \in \mathbb{N}_1$, in this case, system (3.9) reduces to equation (3.5) (**case.3**).

So, we must impose a condition on y_0, which is $y_0 < \dfrac{a_1}{a_2}$ to ensure that $x_1 > 0$.

On the other hand, we have

$$0 < y_{n+1} < \frac{a_4y_n}{a_6y_n} = \frac{a_4}{a_6}, \quad n \in \mathbb{N}_0,$$

that is

$$y_1, y_2, \ldots < \frac{a_4}{a_6}.$$

So, to ensure that $a_1 - a_2 y_n > 0$ and therefore $x_n > 0$, we must also impose $\dfrac{a_4}{a_6} \leq \dfrac{a_1}{a_2}$.

We can conclude that (3.9) can be studied under these two conditions

$$
\begin{aligned}
&0 < y_0 < \frac{a_4}{a_6}, \ \frac{a_4}{a_6} \leq \frac{a_1}{a_2}, \\
&x_0 > 0.
\end{aligned}
$$

Note that in this case

$$x_{n+1} = \frac{a_1 x_n - a_2 x_n y_n}{1 + a_3 x_n} < \frac{a_1 x_n}{1 + a_3 x_n} < \frac{a_1 x_n}{a_3 x_n} = \frac{a_1}{a_3}, \ \ (x_n, y_n > 0)$$

that is, for $y_0 \in \left]0, \dfrac{a_4}{a_6}\right[$, $\left(\dfrac{a_4}{a_6} \leq \dfrac{a_1}{a_2}\right)$, and $x_0 > 0$, we have $0 < y_n < \dfrac{a_4}{a_6}, \ \ \forall n \in \mathbb{N}_0$ and $0 < x_n < \dfrac{a_1}{a_3}, \ \ \forall n = 1, 2, \ldots$

and if $x_0 \in \left]0, \dfrac{a_1}{a_3}\right[: \ \ 0 < x_n < \dfrac{a_1}{a_3}$.

System (3.9) has only one equilibrium point $(\bar{x}, \bar{y})$ in $\left]0, \dfrac{a_1}{a_3}\right[\times \left]0, \dfrac{a_4}{a_6}\right[$, such that

$$
\begin{aligned}
\bar{x} &= \frac{a_6 (a_1 - 1) - a_2 (a_4 - 1)}{a_3 a_6}, \ \ a_6 (a_1 - 1) - a_2 (a_4 - 1) > 0, \\
\bar{y} &= \frac{a_4 - 1}{a_6}, \ \ a_4 > 1.
\end{aligned}
$$

Consider these two continuously differentiable functions

$$h_1 : \left]0, \frac{a_1}{a_3}\right[\times \left]0, \frac{a_4}{a_6}\right[\to \left]0, \frac{a_1}{a_3}\right[$$

$$(x, y) \mapsto h_1 (x, y) = \frac{a_1 x - a_2 x y}{1 + a_3 x},$$

$$h_2 : \left]0, \frac{a_1}{a_3}\right[\times \left]0, \frac{a_4}{a_6}\right[\to \times \left]0, \frac{a_4}{a_6}\right[$$

$$(x, y) \mapsto h_2 (x, y) = \frac{a_4 y}{1 + a_6 y}.$$

So, the Jacobian matrix of the linearized system of (3.9) around $(\bar{x}, \bar{y})$ is given by

$$
\begin{pmatrix}
\dfrac{\partial h_1}{\partial x} (\bar{x}, \bar{y}) & \dfrac{\partial h_1}{\partial y} (\bar{x}, \bar{y}) \\
\dfrac{\partial h_2}{\partial x} (\bar{x}, \bar{y}) & \dfrac{\partial h_2}{\partial y} (\bar{x}, \bar{y})
\end{pmatrix}
$$

$$= \begin{pmatrix} \dfrac{a_6}{a_1a_6 - a_2(a_4-1)} & -\dfrac{a_2(a_6(a_1-1) - a_2(a_4-1))}{a_3(a_1a_6 - a_2(a_4-1))} \\ 0 & \dfrac{1}{a_4} \end{pmatrix}.$$

Additionally, the eigenvalues of the Jacobian matrix around $(\bar{x}, \bar{y})$ are given by
$\lambda_1 = \dfrac{a_6}{a_1a_6 - a_2\,(a_4-1)} < 1,$
$\lambda_2 = \dfrac{1}{a_4} < 1,$
hence, $(\bar{x}, \bar{y}) = \left(\dfrac{a_6\,(a_1-1) - a_2\,(a_4-1)}{a_3a_6}, \dfrac{a_4-1}{a_6}\right)$ is locally asymptotically stable.

Thus, given the previous cases, the choice of conditions $x_0, y_0, z_0 > 0$ in the study of system (3.2) is justified.

Throughout the following, we are going to study system (3.2) with $x_0, y_0, z_0 \in]0, +\infty[$.

Note also that additional conditions will be imposed on x_0, y_0, z_0 and the parameters.

3.3 Dynamical behavior of system (3.2)

This section will closely investigate how the solution to system (3.2) changes and behaves over time.

The following theorem ensures that the solution of system (3.2) is bounded.

Theorem 3.3.1 *Suppose that*

$$0 < x_0 < \frac{a_1}{a_3}, \tag{3.10}$$

$$0 < y_0 < \frac{a_1}{a_2}, \tag{3.11}$$

$$0 < z_0 < \frac{a_7}{a_9} + \frac{a_8a_1}{a_9a_3}, \tag{3.12}$$

and

$$\frac{a_4}{a_6} + \frac{a_5 a_7}{a_6 a_9} + \frac{a_5 a_8 a_1}{a_6 a_9 a_3} < \frac{a_1}{a_2}. \tag{3.13}$$

Then, for every solution $\{(x_n, y_n, z_n)\}_{n \geq 0}$ *to system* (3.2)*, we get*

$$\begin{aligned} x_n &\in I = \left]0, \frac{a_1}{a_3}\right[, \\ y_n &\in J = \left]0, \frac{a_1}{a_2}\right[, \qquad\qquad n \in \mathbb{N}_0. \\ z_n &\in K = \left]0, \frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3}\right[, \end{aligned} \tag{3.14}$$

i.e: *the solution is bounded.*

Proof.

- **n=1**

we have

$$0 < x_1 = \frac{(a_1 - a_2 y_0) x_0}{1 + a_3 x_0} < \frac{(a_1 - a_2 y_0) x_0}{a_3 x_0} = \frac{a_1 - a_2 y_0}{a_3} < \frac{a_1}{a_3},$$

so

$$0 < x_1 < \frac{a_1}{a_3}.$$

Likewise

$$0 < y_1 = \frac{(a_4 + a_5 z_0) y_0}{1 + a_6 y_0} < \frac{(a_4 + a_5 z_0) y_0}{a_6 y_0} = \frac{a_4 + a_5 z_0}{a_6} = \frac{a_4}{a_6} + \frac{a_5}{a_6} z_0,$$

using (3.12), we obtain

$$y_1 < \frac{a_4}{a_6} + \frac{a_5}{a_6}\left(\frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3}\right) = \frac{a_4}{a_6} + \frac{a_5 a_7}{a_6 a_9} + \frac{a_5 a_8 a_1}{a_6 a_9 a_3},$$

using (3.13), we get

$$y_1 < \frac{a_1}{a_2},$$

so

$$0 < y_1 < \frac{a_1}{a_2}.$$

Likewise

$$0 < z_1 = \frac{(a_7 + a_8 x_0) z_0}{1 + a_9 z_0} < \frac{(a_7 + a_8 x_0) z_0}{a_9 z_0} = \frac{a_7 + a_8 x_0}{a_9} = \frac{a_7}{a_9} + \frac{a_8}{a_9} x_0,$$

using (3.10), we obtain

$$z_1 < \frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3},$$

so

$$0 < z_1 < \frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3}.$$

So (3.14) is verified for $n = 1$.

• Suppose that (3.14) is verified at the order n, namely

$$\begin{aligned} x_n &\in I = \left]0, \frac{a_1}{a_3}\right[, \\ y_n &\in J = \left]0, \frac{a_1}{a_2}\right[, \\ z_n &\in K = \left]0, \frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3}\right[. \end{aligned}$$

• We are going to prove its validity at the order $n + 1$.

we have

$$0 < x_{n+1} = \frac{(a_1 - a_2 y_n) x_n}{1 + a_3 x_n} < \frac{(a_1 - a_2 y_n) x_n}{a_3 x_n} = \frac{a_1 - a_2 y_n}{a_3} < \frac{a_1}{a_3},$$

so

$$0 < x_{n+1} < \frac{a_1}{a_3}.$$

Likewise

$$0 < y_{n+1} = \frac{(a_4 + a_5 z_n) y_n}{1 + a_6 y_n} < \frac{(a_4 + a_5 z_n) y_n}{a_6 y_n} = \frac{a_4 + a_5 z_n}{a_6} = \frac{a_4}{a_6} + \frac{a_5}{a_6} z_n,$$

using (3.12), we obtain

$$y_{n+1} < \frac{a_4}{a_6} + \frac{a_5}{a_6}\left(\frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3}\right) = \frac{a_4}{a_6} + \frac{a_5 a_7}{a_6 a_9} + \frac{a_5 a_8 a_1}{a_6 a_9 a_3},$$

using (3.13), we get

$$y_{n+1} < \frac{a_1}{a_2},$$

so

$$0 < y_{n+1} < \frac{a_1}{a_2}.$$

Likewise

$$0 < z_{n+1} = \frac{(a_7 + a_8 x_n)z_n}{1 + a_9 z_n} < \frac{(a_7 + a_8 x_n)z_n}{a_9 z_n} = \frac{a_7 + a_8 x_n}{a_9} = \frac{a_7}{a_9} + \frac{a_8}{a_9}x_n,$$

using (3.10), we obtain

$$z_{n+1} < \frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3},$$

so

$$0 < z_{n+1} < \frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3}.$$

So (3.14) is verified at the order $n+1$, which implying its validity for all $n \geq 0$. ■

3.3.1 Local stability

Here, we are going to investigate the local stability of the equilibrium point of system (3.2).

Consider three functions, f, g, and h, all of which are continuously differentiable, such that

$$f : I \times J \times K \longrightarrow I,$$

$$g : I \times J \times K \longrightarrow J,$$

$$h : I \times J \times K \longrightarrow K,$$

$I = \left]0, \frac{a_1}{a_3}\right[$, $J = \left]0, \frac{a_1}{a_2}\right[$, and $K = \left]0, \frac{a_7}{a_9} + \frac{a_8 a_1}{a_9 a_3}\right[$.

Let's examine the following difference equations system

$$\begin{cases} x_{n+1} = f(x_n, y_n, z_n), \\ y_{n+1} = g(x_n, y_n, z_n), \\ z_{n+1} = h(x_n, y_n, z_n), \end{cases} \tag{3.15}$$

with $n \in \mathbb{N}_0$ and $(x_0, y_0, z_0) \in I \times J \times K$.

An equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ for system (3.15), is characterized as a solution of the following system

$$\begin{cases} \bar{x} = f(\bar{x}, \bar{y}, \bar{z}), \\ \bar{y} = g(\bar{x}, \bar{y}, \bar{z}), \\ \bar{z} = h(\bar{x}, \bar{y}, \bar{z}). \end{cases} \tag{3.16}$$

From where, if $(\bar{x}, \bar{y}, \bar{z})$ constitutes an equilibrium point in system (3.2), it satisfies

$$\begin{cases} \bar{x} &= \dfrac{a_1\bar{x} - a_2\bar{x}\bar{y}}{1 + a_3\bar{x}}, \\ \\ \bar{y} &= \dfrac{a_4\bar{y} + a_5\bar{y}\bar{z}}{1 + a_6\bar{y}}, \\ \\ \bar{z} &= \dfrac{a_7\bar{z} + a_8\bar{z}\bar{x}}{1 + a_9\bar{z}}. \end{cases}$$

The lemma below outlines the equilibrium point of system (3.2).

Lemma 3.3.1 *Let* $P = \left(\frac{L}{S}, \frac{M}{S}, \frac{N}{S}\right)$, *such that*

$$\begin{aligned} L &= a_6 a_9 (a_1 - 1) - a_2 (a_9 (a_4 - 1) + a_5 (a_7 - 1)), \\ M &= a_3 a_9 (a_4 - 1) + a_5 (a_3 (a_7 - 1) + a_8 (a_1 - 1)), \\ N &= a_3 a_6 (a_7 - 1) + a_8 (a_6 (a_1 - 1) - a_2 (a_4 - 1)), \\ &\textit{and} \\ S &= a_2 a_5 a_8 + a_3 a_6 a_9. \end{aligned}$$

If

$$a_6a_9 > \frac{a_2\left(a_9\left(a_4-1\right)+a_5\left(a_7-1\right)\right)}{a_1-1},\ \ a_1>1,\ \ a_4>1,\ \ a_7>1,\ \ a_6>\frac{a_2\left(a_4-1\right)}{a_1-1}, \tag{3.17}$$

is verified, so, P is the unique equilibrium point of system (3.2).

• Note that condition (3.17) ensures that $P=\left(\frac{L}{S},\frac{M}{S},\frac{N}{S}\right)\in\left]0,\frac{a_1}{a_3}\right[\times\left]0,\frac{a_1}{a_2}\right[\times\left]0,\frac{a_7}{a_9}+\frac{a_8a_1}{a_9a_3}\right[$.

The theorem below asserts the local stability of the equilibrium of system (3.2).

Theorem 3.3.2 *Suppose that the statement* (3.17) *is held.*

then, P is locally asymptotically stable if

$$\Psi < \left(S+a_3L\right)^2\left(S+a_6M\right)^2\left(S+a_9N\right)^2. \tag{3.18}$$

Where

$$\begin{aligned}
\Psi &= S^3\left(a_1S+a_2M\right)\left(a_4S+a_5N\right)\left(a_7S+a_8L\right)\\
&+ S^2\Big[\left(a_4S+a_5N\right)\left(a_7S+a_8L\right)\left(S+a_3L\right)^2+\left(a_1S+a_2M\right)\left(a_7S+a_8L\right)\left(S+a_6M\right)^2\\
&+ \left(a_1S+a_2M\right)\left(a_4S+a_5N\right)\left(S+a_9N\right)^2\Big]+S\Big[\left(a_1S+a_2M\right)\left(S+a_6M\right)^2\left(S+a_9N\right)^2\\
&+ \left(a_4S+a_5N\right)\left(S+a_3L\right)^2\left(S+a_9N\right)^2+\left(a_7S+a_8L\right)\left(S+a_3L\right)^2\left(S+a_6M\right)^2\Big]\\
&+ a_2a_5a_8LMN\left(S+a_3L\right)\left(S+a_6M\right)\left(S+a_9N\right).
\end{aligned}$$

Proof. Assume the statement (3.17) is held.

The characteristic polynomial of the Jacobian matrix around $P=\left(\frac{L}{S},\frac{M}{S},\frac{N}{S}\right)$ is given by

$$\Upsilon(\lambda)=-\lambda^3+\lambda^2\left(A_1-A_2+A_3+A_4\right)-\lambda\left(A_5-A_6+A_7-A_8+A_9\right)+A_{10}-A_{11}-A_{12},$$

where

$$A_1=\frac{a_1S^2}{\left(S+a_3L\right)^2},\quad A_2=\frac{a_2MS}{\left(S+a_3L\right)^2},\quad A_3=\frac{S\left(a_4S+a_5N\right)}{\left(S+a_6M\right)^2},\quad A_4=\frac{S\left(a_7S+a_8L\right)}{\left(S+a_9N\right)^2},$$

$$
\begin{aligned}
A_5 &= \frac{a_1 S^3 (a_4 S + a_5 N)}{(S + a_3 L)^2 (S + a_6 M)^2}, \\
A_6 &= \frac{a_2 M S^2 (a_4 S + a_5 N)}{(S + a_3 L)^2 (S + a_6 M)^2}, \\
A_7 &= \frac{a_1 S^3 (a_7 S + a_8 L)}{(S + a_3 L)^2 (S + a_9 N)^2}, \\
A_8 &= \frac{a_2 M S^2 (a_7 S + a_8 L)}{(S + a_3 L)^2 (S + a_9 N)^2}, \\
A_9 &= \frac{S^2 (a_4 S + a_5 N)(a_7 S + a_8 L)}{(S + a_6 M)^2 (S + a_9 N)^2}, \\
A_{10} &= \frac{a_1 S^4 (a_4 S + a_5 N)(a_7 S + a_8 L)}{(S + a_3 L)^2 (S + a_6 M)^2 (S + a_9 N)^2}, \\
A_{11} &= \frac{a_2 M S^3 (a_4 S + a_5 N)(a_7 S + a_8 L)}{(S + a_3 L)^2 (S + a_6 M)^2 (S + a_9 N)^2},
\end{aligned}
$$

and

$$
A_{12} = \frac{a_2 a_5 a_8 LMN}{(S + a_3 L)(S + a_6 M)(S + a_9 N)}.
$$

Let's put

$$R(\lambda) = -\lambda^3,$$

and

$$T(\lambda) = -\lambda^2 (A_1 - A_2 + A_3 + A_4) + \lambda (A_5 - A_6 + A_7 - A_8 + A_9) - A_{10} + A_{11} + A_{12}.$$

Assume that

$$\Psi < (S + a_3 L)^2 (S + a_6 M)^2 (S + a_9 N)^2,$$

then, for $|\lambda| = 1$, we get

$$
\begin{aligned}
|T(\lambda)| &\le \sum_{i=1}^{12} A_i \\
&= \frac{\Psi}{(S + a_3 L)^2 (S + a_6 M)^2 (S + a_9 N)^2} \\
&< 1 = |R(\lambda)|.
\end{aligned}
$$

Then, according to Rouche's theorem, $R(\lambda)$ and $R(\lambda) - T(\lambda)$ have the same number of zeroes within the open unit disk $|\lambda| < 1$. Hence, P is locally asymptotically stable. ■

3.3.2 Global stability

Here, we are going to examine the global stability of the equilibrium point of system (3.2).

The following theorem presents the convergence of the positive solution $\{(x_n, y_n, z_n)\}_{n\geq 0}$ of system (3.2) to the equilibrium point.

Theorem 3.3.3 *Assume that* (3.17) *holds and* $a_3a_6a_9 - a_2a_5a_8 > 0$, *then the equilibrium point* P *of system* (3.2) *is a global attractor.*

Proof. Consider system (3.2) with the initial values $(x_0, y_0, z_0) \in I \times J \times K$,

Let's put

$f : I \times J \times K \to I$

$$(x, y, z) \mapsto f(x, y, z) = \frac{a_1x - a_2xy}{1 + a_3x},$$

$g : I \times J \times K \to J$

$$(x, y, z) \mapsto g(x, y, z) = \frac{a_4y + a_5yz}{1 + a_6y},$$

$h : I \times J \times K \to K$

$$(x, y, z) \mapsto h(x, y, z) = \frac{a_7z + a_8zx}{1 + a_9z},$$

where I, J and K are three positive real intervals respectively given by $\left]0, \frac{a_1}{a_3}\right[$, $\left]0, \frac{a_1}{a_2}\right[$, and $\left]0, \frac{a_7}{a_9} + \frac{a_8a_1}{a_9a_3}\right[$.

We know that $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are bounded, so there exist m_1, M_1, m_2, M_2, m_3 and M_3, such that

$$\begin{aligned} m_1 &= \lim_{n\to\infty} \inf x_n, \quad M_1 = \lim_{n\to\infty} \sup x_n, \\ m_2 &= \lim_{n\to\infty} \inf y_n, \quad M_2 = \lim_{n\to\infty} \sup y_n, \\ m_3 &= \lim_{n\to\infty} \inf z_n, \quad M_3 = \lim_{n\to\infty} \sup z_n. \end{aligned} \tag{3.19}$$

Using the definition of lim inf and lim sup, we obtain

$$\begin{aligned}
&\forall \epsilon_1 \in]0, m_1[\,, \exists n_1 \in \mathbb{N}_0, \forall n \geq n_1 : m_1 - \epsilon_1 \leq x_n \leq M_1 + \epsilon_1,\\
&\forall \epsilon_2 \in]0, m_2[\,, \exists n_2 \in \mathbb{N}_0, \forall n \geq n_2 : m_2 - \epsilon_2 \leq y_n \leq M_2 + \epsilon_2,\\
&\forall \epsilon_3 \in]0, m_3[\,, \exists n_3 \in \mathbb{N}_0, \forall n \geq n_3 : m_3 - \epsilon_3 \leq z_n \leq M_3 + \epsilon_3.
\end{aligned} \tag{3.20}$$

Let put $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3)$ and $n_0 = \max(n_1, n_2, n_3)$.

It is easy to see that f is increasing in x and z and decreasing in y, so

$$\begin{aligned}
&f(m_1 - \epsilon, y_n, z_n) \leq f(x_n, y_n, z_n) \leq f(M_1 + \epsilon, y_n, z_n),\\
&f(m_1 - \epsilon, M_2 + \epsilon, z_n) \leq f(x_n, y_n, z_n) \leq f(M_1 + \epsilon, m_2 - \epsilon, z_n),\\
&f(m_1 - \epsilon, M_2 + \epsilon, m_3 - \epsilon) \leq f(x_n, y_n, z_n) \leq f(M_1 + \epsilon, m_2 - \epsilon, M_3 + \epsilon),\\
&f(m_1 - \epsilon, M_2 + \epsilon, m_3 - \epsilon) \leq m_1 \leq M_1 \leq f(M_1 + \epsilon, m_2 - \epsilon, M_3 + \epsilon),
\end{aligned}$$

by passing to the limit when $\epsilon \to 0$ (take in consideration that f is continuous), we obtain

$$f(m_1, M_2, m_3) \leq m_1 \leq M_1 \leq f(M_1, m_2, M_3). \tag{3.21}$$

From (3.21), we get

$$\begin{aligned}
f(m_1, M_2, m_3) \leq m_1 \quad &\Leftrightarrow f(m_1, M_2, m_3) - m_1 \leq 0\\
&\Leftrightarrow \frac{a_1 m_1 - a_2 m_1 M_2}{1 + a_3 m_1} - m_1 \leq 0\\
&\Leftrightarrow \frac{a_1 - a_2 M_2}{1 + a_3 m_1} - 1 \leq 0
\end{aligned}$$

that is to say

$$a_1 - a_2 M_2 \leq 1 + a_3 m_1. \tag{3.22}$$

We get also from (3.21)

$$\begin{aligned}
M_1 \leq f(M_1, m_2, M_3) \quad &\Leftrightarrow M_1 - f(M_1, m_2, M_3) \leq 0\\
&\Leftrightarrow M_1 - \frac{a_1 M_1 - a_2 M_1 m_2}{1 + a_3 M_1} \leq 0,\\
&\Leftrightarrow 1 - \frac{a_1 - a_2 m_2}{1 + a_3 M_1} \leq 0
\end{aligned}$$

that is to say

$$a_2 m_2 - a_1 \leq -1 - a_3 M_1. \tag{3.23}$$

From (3.22) and (3.23), we get

$$a_3(M_1 - m_1) \leq a_2(M_2 - m_2). \tag{3.24}$$

Likewise, using the fact that g is increasing in all arguments, we get

$$g(m_1, m_2, m_3) \leq m_2 \leq M_2 \leq g(M_1, M_2, M_3). \tag{3.25}$$

From (3.25), we get

$$\begin{aligned} g(m_1, m_2, m_3) \leq m_2 \quad &\Leftrightarrow g(m_1, m_2, m_3) - m_2 \leq 0 \\ &\Leftrightarrow \frac{a_4 m_2 + a_5 m_2 m_3}{1 + a_6 m_2} - m_2 \leq 0 \\ &\Leftrightarrow \frac{a_4 + a_5 m_3}{1 + a_6 m_2} - 1 \leq 0 \end{aligned}$$

that is to say

$$a_4 + a_5 m_3 \leq 1 + a_6 m_2. \tag{3.26}$$

We get also from (3.25)

$$\begin{aligned} M_2 \leq g(M_1, M_2, M_3) \quad &\Leftrightarrow M_2 - g(M_1, M_2, M_3) \leq 0 \\ &\Leftrightarrow M_2 - \frac{a_4 M_2 + a_5 M_2 M_3}{1 + a_6 M_2} \leq 0, \\ &\Leftrightarrow 1 - \frac{a_4 + a_5 M_3}{1 + a_6 M_2} \leq 0 \end{aligned}$$

that is to say

$$1 + a_6 M_2 \leq a_4 + a_5 M_3. \tag{3.27}$$

From (3.26) and (3.27), we get

$$a_6(M_2 - m_2) \leq a_5(M_3 - m_3). \tag{3.28}$$

Now, using the fact that h is increasing in all arguments, we get

$$h(m_1, m_2, m_3) \leq m_3 \leq M_3 \leq h(M_1, M_2, M_3). \tag{3.29}$$

From (3.29), we get

$$\begin{aligned} h\left(m_1,m_2,m_3\right) \leq m_3 \quad & \Leftrightarrow h\left(m_1,m_2,m_3\right) - m_3 \leq 0 \\ & \Leftrightarrow \frac{a_7 m_3 + a_8 m_3 m_1}{1 + a_9 m_3} - m_3 \leq 0 \\ & \Leftrightarrow \frac{a_7 + a_8 m_1}{1 + a_9 m_3} - 1 \leq 0 \end{aligned}$$

that is to say

$$a_7 + a_8 m_1 \leq 1 + a_9 m_3. \tag{3.30}$$

We get also from (3.29)

$$\begin{aligned} M_3 \leq h\left(M_1,M_2,M_3\right) \quad & \Leftrightarrow M_3 \leq \frac{a_7 M_3 + a_8 M_3 M_1}{1 + a_9 M_3} \\ & \Leftrightarrow M_3 - \frac{a_7 M_3 + a_8 M_3 M_1}{1 + a_9 M_3} \leq 0, \\ & \Leftrightarrow 1 - \frac{a_7 + a_8 M_1}{1 + a_9 M_3} \leq 0 \end{aligned}$$

that is to say

$$a_7 + a_8 M_1 \leq 1 + a_9 M_3. \tag{3.31}$$

From (3.30) and (3.31), we get

$$a_9\left(M_3 - m_3\right) \leq a_8\left(M_1 - m_1\right). \tag{3.32}$$

Multiplying (3.32) by a_3, we get

$$a_3 a_9\left(M_3 - m_3\right) \leq a_3 a_8\left(M_1 - m_1\right).$$

Using (3.24), we obtain

$$a_3 a_9\left(M_3 - m_3\right) \leq a_8 a_2\left(M_2 - m_2\right),$$

multiplying by a_6, we get

$$a_6 a_3 a_9\left(M_3 - m_3\right) \leq a_6 a_8 a_2\left(M_2 - m_2\right).$$

Using (3.28), we obtain

$$a_6a_3a_9\left(M_3-m_3\right)\leq a_2a_8a_5\left(M_3-m_3\right),$$

so

$$\left(a_6a_3a_9-a_2a_8a_5\right)\left(M_3-m_3\right)\leq 0.$$

Since $a_3a_6a_9-a_2a_5a_8>0$, so $M_3-m_3\leq 0$, from where $m_3=M_3$.

Theorem 3.3.4 *Suppose that (3.17) and (3.18) hold. If $a_3a_6a_9-a_2a_5a_8>0$, Then, P is globally asymptotically stable.*

Proof. The proof is derived from theorem (3.3.2) and theorem (3.3.3). ■

To validate these theoretical findings, we are going to consider the following numerical example.

Example 3.3.1
- *let $a_1=2$, $a_2=2$, $a_3=6$, $a_4=2$, $a_5=3$, $a_6=4$, $a_7=2$, $a_8=1$ and $a_9=6$ in system (3.2), so we obtain the following system with the previous parameters that comply with (3.17) and (3.18), and that verify $a_3a_6a_9-a_2a_5a_8>0$*

$$x_{n+1}=\frac{2x_n-2x_ny_n}{1+6x_n},\quad y_{n+1}=\frac{2y_n+3y_nz_n}{1+4y_n},\quad z_{n+1}=\frac{2z_n+z_nx_n}{1+6z_n}. \tag{3.33}$$

Suppose that

$$x_0=\frac{1}{4},\quad y_0=\frac{1}{3}\ \text{and}\ z_0=\frac{1}{3}, \tag{3.34}$$

so, the equilibrium point $P=\left(\frac{1}{25},\frac{19}{50},\frac{13}{75}\right)$ of system (3.33) is globally asymptotically stable, and we get the graph in Fig (3.2).

3.3.3 Rate of convergence

In this section, we are going to delve into exploring the rate of convergence of any solution that converges to the equilibrium point $P=\left(\frac{L}{S},\frac{M}{S},\frac{N}{S}\right)$ of system (3.2).

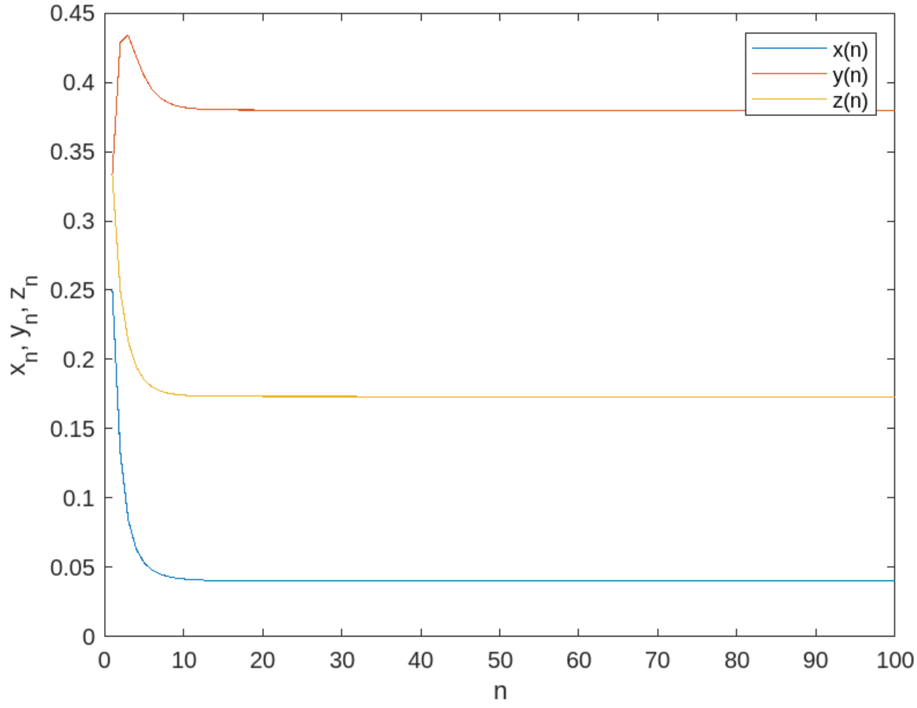

Figure 3.2: Plot of the solution to system (3.33) with the initial values (3.34).

Consider $\{(x_n, y_n, z_n)\}_{n\geq 0}$ as a solution of system (3.2), such that

$$\lim_{n\to\infty} x_n = \bar{x}, \quad \lim_{n\to\infty} y_n = \bar{y} \text{ and } \lim_{n\to\infty} z_n = \bar{z},$$

where

$$(\bar{x}, \bar{y}, \bar{z}) = P.$$

To find the error terms, we get from system (3.2)

$$\begin{aligned}
x_{n+1} - \bar{x} &= \frac{a_1 x_n - a_2 x_n y_n}{1 + a_3 x_n} - \frac{a_1 \bar{x} - a_2 \bar{x}\bar{y}}{1 + a_3 \bar{x}} \\
&= \frac{(a_1 - a_2 y_n)}{(1 + a_3 x_n)(1 + a_3 \bar{x})} (x_n - \bar{x}) - \frac{a_2 \bar{x}}{1 + a_3 \bar{x}} (y_n - \bar{y}), \\
y_{n+1} - \bar{y} &= \frac{a_4 y_n + a_5 y_n z_n}{1 + a_6 y_n} - \frac{a_4 \bar{y} + a_5 \bar{y}\bar{z}}{1 + a_6 \bar{y}} \\
&= \frac{(a_4 + a_5 z_n)}{(1 + a_6 y_n)(1 + a_6 \bar{y})} (y_n - \bar{y}) + \frac{a_5 \bar{y}}{1 + a_6 \bar{y}} (z_n - \bar{z}),
\end{aligned}$$

and

$$\begin{aligned}
z_{n+1} - \bar{z} &= \frac{a_7 z_n + a_8 z_n x_n}{1 + a_9 z_n} - \frac{a_7 \bar{z} + a_8 \bar{z}\bar{x}}{1 + a_9 \bar{z}} \\
&= \frac{(a_7 + a_8 x_n)}{(1 + a_9 z_n)(1 + a_9 \bar{z})} (z_n - \bar{z}) + \frac{a_8 \bar{z}}{1 + a_9 \bar{z}} (x_n - \bar{x}).
\end{aligned}$$

For $n \geq 0$, we put

$$e_n^1 = x_n - \bar{x}, \;\; e_n^2 = y_n - \bar{y} \;\text{ and }\; e_n^3 = z_n - \bar{z},$$

then, the previous equalities can be written as follow

$$e_{n+1}^1 = a_n e_n^1 + b_n e_n^2, \;\; e_{n+1}^2 = c_n e_n^2 + d_n e_n^3 \;\text{ and }\; e_{n+1}^3 = s_n e_n^3 + r_n e_n^1,$$

where

$$\begin{aligned} a_n &= \frac{(a_1 - a_2 y_n)}{(1 + a_3 x_n)(1 + a_3 \bar{x})}, \;\; b_n = -\frac{a_2 \bar{x}}{1 + a_3 \bar{x}}, \\ c_n &= \frac{(a_4 + a_5 z_n)}{(1 + a_6 y_n)(1 + a_6 \bar{y})}, \;\; d_n = \frac{a_5 \bar{y}}{1 + a_6 \bar{y}}, \\ s_n &= \frac{(a_7 + a_8 x_n)}{(1 + a_9 z_n)(1 + a_9 \bar{z})}, \;\; r_n = \frac{a_8 \bar{z}}{1 + a_9 \bar{z}}. \end{aligned}$$

So, we can write

$$\begin{aligned} a_n &= a + \alpha_n, \quad b_n = b + \beta_n, \\ c_n &= c + \gamma_n, \quad d_n = d + \delta_n, \\ s_n &= s + \sigma_n, \quad r_n = r + \rho_n, \end{aligned}$$

such that

$$\begin{aligned} a &= \frac{(a_1 - a_2 \bar{y})}{(1 + a_3 \bar{x})^2}, \;\; b = -\frac{a_2 \bar{x}}{1 + a_3 \bar{x}}, \\ c &= \frac{(a_4 + a_5 \bar{z})}{(1 + a_6 \bar{y})^2}, \;\; d = \frac{a_5 \bar{y}}{1 + a_6 \bar{y}}, \\ s &= \frac{(a_7 + a_8 \bar{x})}{(1 + a_9 \bar{z})^2}, \;\; r = \frac{a_8 \bar{z}}{1 + a_9 \bar{z}}, \end{aligned}$$

and

$$\alpha_n = \frac{-a_1 a_3 (x_n - \bar{x}) - a_2 (y_n - \bar{y}) + a_2 a_3 (x_n \bar{y} - \bar{x} y_n)}{(1 + a_3 x_n)(1 + a_3 \bar{x})^2}, \quad \beta_n = 0,$$

$$\gamma_n = \frac{-a_4 a_6 (y_n - \bar{y}) + a_5 (z_n - \bar{z}) - a_5 a_6 (y_n \bar{z} - \bar{y} z_n)}{(1 + a_6 y_n)(1 + a_6 \bar{y})^2}, \quad \delta_n = 0,$$

$$\sigma_n = \frac{-a_7 a_9 (z_n - \bar{z}) + a_8 (x_n - \bar{x}) - a_8 a_9 (z_n \bar{x} - \bar{z} x_n)}{(1 + a_9 z_n)(1 + a_9 \bar{z})^2}, \quad \rho_n = 0.$$

Since

$$\lim_{n\to\infty} x_n = \bar{x}, \ \lim_{n\to\infty} y_n = \bar{y} \ \text{and} \ \lim_{n\to\infty} z_n = \bar{z},$$

then

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \rho_n = 0.$$

The error system is given by

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \\ e_{n+1}^3 \end{pmatrix} = \left[\begin{pmatrix} a & b & 0 \\ 0 & c & d \\ r & 0 & s \end{pmatrix} + \begin{pmatrix} \alpha_n & \beta_n & 0 \\ 0 & \gamma_n & \delta_n \\ \rho_n & 0 & \sigma_n \end{pmatrix} \right] \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_n^3 \end{pmatrix},$$

that is

$$X_{n+1} = (A + B_n) X_n, \ \ n \in \mathbb{N}_0,$$

where

$$X_n = \left(e_n^1, e_n^2, e_n^3\right)^T,$$

the constant matrix A is of the form

$$A = \begin{pmatrix} a & b & 0 \\ 0 & c & d \\ r & 0 & s \end{pmatrix},$$

$$= \begin{pmatrix} \dfrac{a_1 - a_2\bar{y}}{(1+a_3\bar{x})^2} & -\dfrac{a_2\bar{x}}{1+a_3\bar{x}} & 0 \\ 0 & \dfrac{a_4 + a_5\bar{z}}{(1+a_6\bar{y})^2} & \dfrac{a_5\bar{y}}{1+a_6\bar{y}} \\ \dfrac{a_8\bar{z}}{1+a_9\bar{z}} & 0 & \dfrac{a_7 + a_8\bar{x}}{(1+a_9\bar{z})^2} \end{pmatrix},$$

and

$$B_n = \begin{pmatrix} \alpha_n & \beta_n & 0 \\ 0 & \gamma_n & \delta_n \\ \rho_n & 0 & \sigma_n \end{pmatrix},$$

with $\|B_n\| \to 0$ when $n \to \infty$.

Using propositions (1.1.1) and (1.1.2), we obtain the following result.

Theorem 3.3.5 *Suppose $\{(x_n, y_n, z_n)\}_{n\geq 0}$ is a positive solution of system (3.2), that satisfies*

$$\lim_{n\to\infty} x_n = \bar{x}, \ \lim_{n\to\infty} y_n = \bar{y} \ \text{and} \ \lim_{n\to\infty} z_n = \bar{z},$$

where

$$(\bar{x}, \bar{y}, \bar{z}) = P.$$

So, the error vector $e_n = \left(e_n^1, e_n^2, e_n^3\right)^T$ of every solution of system (3.2) meets both of the asymptotic relations below

$$\lim_{n\to\infty} (\|e_n\|)^{\frac{1}{n}} = |\lambda_{1,2,3} J_F(\bar{x}, \bar{y}, \bar{z})|, \quad \lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_{1,2,3} J_F(\bar{x}, \bar{y}, \bar{z})|,$$

with $\lambda_{1,2,3} J_F(\bar{x}, \bar{y}, \bar{z})$ is a characteristic root of the Jacobian matrix $J_F(\bar{x}, \bar{y}, \bar{z})$.

General conclusion and outlook

This thesis is a detailed summary of various research studies that looked at the form of solutions and how these solutions behave in specific systems of nonlinear difference equations. By carefully analyzing and investigating these systems, it aims to explain the complex patterns and changes seen in the solutions, providing valuable information about their traits and properties.

In the first chapter, we gave the solutions to the following $k-$dimensional close-to-cyclic nonlinear difference equations system

$$y_{n+1}^{(i)} = \frac{a_i y_n^{(i+1)} \left(y_{n-k}^{(i+1)}\right)^{p_{i+1}} + b_i}{\left(y_{n-k+1}^{(i)}\right)^{p_i}}; \quad n \in \mathbb{N}_0,$$

where $y_n^{(i+k)} = y_n^{(i)}$, $p_{i+k} = p_i$, $a_{i+k} = a_i$, $b_{i+k} = b_i$, $i = \overline{1,k}$, the initial values $y_{-k}^{(i)}, y_{-k+1}^{(i)}, \ldots, y_0^{(i)}$ and the parameters a_i and b_i, $i = \overline{1,k}$ are positive real numbers and p_i, $i = \overline{1,k}$, are real numbers. We also examined the asymptotic behavior of the the equilibrium point in special cases.

In the second chapter, we studied the following symmetric higher-order difference equations system

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + y_{n-k}}, \quad y_{n+1} = \frac{y_{n-(2k+1)}}{1 + x_{n-k}}, \quad n, k \in \mathbb{N}_0,$$

the initial values $x_{-(2k+1)}, x_{-2k}, \ldots, x_0, y_{-(2k+1)}, y_{-2k}, \ldots, y_0$ are non-negative real numbers. We also combined its properties into a very important theorem.

In the near future, we will try to generalize the previous system to the following close-to-symmetric one

$$x_{n+1} = \frac{x_{n-(2k+1)}}{\alpha + y_{n-k}}, \quad y_{n+1} = \frac{y_{n-(2k+1)}}{\beta + x_{n-k}}, \quad n, k \in \mathbb{N}_0,$$

the initial values $x_{-(2k+1)}, x_{-2k}, \ldots, x_0, y_{-(2k+1)}, y_{-2k}, \ldots, y_0$, and the parameters α and β are positive real numbers.

In the third chapter, we studied this nonlinear difference equations system

$$x_{n+1} = \frac{a_1 x_n - a_2 x_n y_n}{1 + a_3 x_n}, \quad y_{n+1} = \frac{a_4 y_n + a_5 y_n z_n}{1 + a_6 y_n}, \quad z_{n+1} = \frac{a_7 z_n + a_8 z_n x_n}{1 + a_9 z_n}, \quad n \in \mathbb{N}_0,$$

where the parameters a_i, $i = \overline{1, 9}$ and the initial values x_0, y_0 and z_0 are positive real numbers. We also investigated the local stability of its equilibrium point, and studied the asymptotic behavior of this equilibrium.

In the near future, we will try to generalize the previous system to the following $P-$dimensional one

$$x^{(i)}_{n+1} = \frac{a_i x^{(i)}_n + b_i x^{(i)}_n x^{(i+1)}_n}{1 + c_i x^{(i)}_n}; \quad n \in \mathbb{N}_0,$$

where $x^{(i+P)}_n = x^{(i)}_n$, $a_{i+P} = a_i$, $b_{i+P} = b_i$ and $c_{i+P} = c_i$, $i = \overline{1, P}$, the initial values $x^{(i)}_0$ and the parameters a_i and c_i, $i = \overline{1, P}$ present positive real numbers and the parameters b_i, $i = \overline{1, P}$ are nonzero real numbers.

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