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Application of reproducing kernel Hilbert space method to some ordinary Differential equations of fractional order

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Dedication

Sara

This work is dedicated to

my father, who did not deprive me of anything, and my mother woh provided me with tenderness, may God have mercy on her, my sisters and my brother,

my whole family and my colleague at work Manel .

Manel

This work is dedicated to the owner of a fragrant biography and enlightened thought, for ho had the first credit in attaining higher education (my beloved father), may god prolong his life.

To those who set me on the path of life, made me calm, and took care of me until became old (my mother) and my friend Sara.

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$\mathscr{H}$: Hilbert Space.

$L^p[a,b]$: Lebesgue Spaces.

$\Gamma(\alpha)$: Gamma function with parameter α.

$B(\alpha,\beta)$: Beta function with parameters α and β.

$\mathscr{I}_a^n$: Riemann-Liouville fractional integral of order n.

$\mathscr{D}_a^n$: Riemann-Liouville fractional derivative of order n.

$\mathscr{D}_{*a}^n$: Caputo fractional derivative of order n.

$\mathcal{A}^m[a,b]$: Set of functions with absolutely continuous derivative of order m-1.

$\mathscr{W}_2^m[a,b]$: Sobolev Space.

$\mathrm{k}(x,y)$: Reproducing kernel function.

$\{\Psi_i\}_{i=1}^{\infty}$: Orthogonal functions system.

$\left\{\overline{\Psi}_i\right\}_{i=1}^{\infty}$: Orthonormal functions system.

B_{ik} : Orthogonalization coefficients.

$\mathcal{L}$: Bounded linear operator.

Abbreviations

$Abs.C$: Absolutely Continuous.

$BC's$: Boundary Conditions.

$BVPs$: Boundary Value Problems.

FDE : Fractional Differential Equations.

$RKHS$: Reproducing Kernel Hilbert Space.

Introduction

The first definition of the fractional derivative was presented by Liouville and Riemann at the end of the 19th century, however, the notion of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus was mentioned already in 1695 by Leibniz and LHospital.

Fractional differential equations (FDEs) have recently been discovered in a variety of disciplines, including physics, chemistry, and engineering [21,25]. Since most fractional differential equations do not have exact analytical solutions, approximation and numerical techniques are used such as the Adomians decomposition method (ADM) [2], variational iteration method (VIM) [22], differential transform method [15] and homotopy perturbation method (HPM) [1] have been used for solving a wide range of problems.

One of the numerical methods that has been applied to solve fractional differential equations is the reproducing kernel Hilbert space method (RKHS). Three mathematicians from Berlin initially introduced the replicating kernels concept: (Szeg, 1921), (Bergman, 1922), and (Bochner, 1922).

In 1935, the positive definite kernels was examined by E, Moore in his general analysis named positive Hermitian mateix. N. Aronszajn used the term "Reproducing kernel function" in 1950 and established the existence and uniqueness of a reproducing kernel Hilbert space. Cui proved in 1986 that $\mathscr{W}_2^1[a,b]$ is a Hilbert space with a reproducing kernel function that can be stated by a finite term, and as a result, the application of reproducing kernel theory started to spread throughout various fields. S. Saitoh presented the general theory of reproducing kernel Hilbert spaces and its numerous applications in 1988.

Numerous researchers have used the RKHS method in recent years to obtain the analytical approximate solutions to many problems, including regular and singular initial value problems (IVPs), regular, singular, singular weakly, singular periodic and singularly perturbed boundary value problems BVPs, system of regular and singular IVPs and BVPs, regular and singular integral equations (IEs), partial differential equations (PDE) and inverse problems in PDEs. the Reproducing kernel theory also has a significant role in statistics and probability [6].

In this thesis, we apply the RKHS method to give approximate solutions for linear and nonlinear differential equations of fractional order. The numerical results illustrate that the method is quite accurate and efficient for solving fractional differential equations. The analytical solution is represented in the form of series in the reproducing kernel space and the approximate solution $u_n(x)$ is obtained by the $n-$term intercept of the analytical solution and is proved to converge to the analytical solution.

This thesis is organized as follows: In Chapter one, we introduce some fundamental concepts and definitions of functional analysis and fractional calculus. In Chapter Two, we give basic concepts, definition and theorems of RKHS, then we present the reproducing kernel function by re-defining the inner product of a reproducing kernel space in order to obtain the analytical approximate solution for a general form of ordinary differential equations (ODEs). Also, we describe the analysis of the RKHS method and introduce an effective algorithm based on this method. In Chapter three, we apply the RKHSM to approximate the solution of IVP fractional differential equations of first order, IVP and BVP fractional differential equations of second order. Various numerical example are presented to illustrate the efficiency and the accuracy of the method. The thesis end in chapter four with conclusion and future recommendations.

Chapter 1

Preliminaries

In this chapter, some basic concepts in functional analysis and fractional calculus will be presented.

1.1 Functional Analysis

The study of vector spaces that possess a limit-related structure such as inner product, norm, topology, etc., and the linear functions that satisfy these structures in a suitable manner is the essence of functional analysis.

The majority of the definitions and characteristics in this section have been obtained from [17].

1.1.1 Normed Spaces

Definition 1.1.1. *Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A norm on X is a function $\|.\| : X \longrightarrow [0, \infty)$ such that*

1. *For all $x \in X$, $\|x\| \geq 0$, if $x \in X$, then $\|x\| = 0$ iff $x = 0$ (Positive definite),*
2. *For all $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and for all $x \in X$, $\|\alpha x\| = |\alpha| \, \|x\|$,*

3. *For all* $x,\ y \in X$, $\|x+y\| \leq \|x\| + \|y\|$ *(Triangle inequality).*

Definition 1.1.2. *A vector space with norm defined on it is called* ***normed space****.*

Example 1.1.1. 1. $\mathbb{R}$ *is a vector space over* $\mathbb{R}$*, and if we define* $\|.\| : \mathbb{R} \longrightarrow [0,\infty)$ *by* $\|x\| = |x|$, $x \in \mathbb{R}$*, then it becomes a normed space.*

2. $\mathbb{R}^n$ *is a vector space over* $\mathbb{R}$*, and let* $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$, $x = [x_1, \ldots, x_n] \in \mathbb{R}^n$*, then* $\mathbb{R}^n$ *is a normed space.*

In a normed space, we possess a concept of "distance" between vectors, and we can state when two vectors are approximate or distant. Hence, we can talk about convergent sequences and Cauchy sequences in a normed space.

Definition 1.1.3. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* X*, and let* $x \in X$*, the sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* x *if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ \ \textit{satisfying}\ \ n \geq N,\ \|x_n - x\| \leq \varepsilon. \qquad (1.1)$$

Note that (1.1) says that the real sequence $(\|x_n - x\|)_{n\in\mathbb{N}}$ converges to 0, i.e

$$lim_{n\to\infty} \|x_n - x\| = 0.$$

Example 1.1.2. *The sequence* $(x_n)_{n\in\mathbb{N}}$ *is called a Cauchy sequence if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ \ \textit{such that}\ \forall\ m,\ n \in \mathbb{N}\ \ \textit{satisfying}\ \ m,\ n \geq N,\ \|x_m - x_n\| < \varepsilon. \qquad (1.2)$$

Remark 1.1.1. *Every convergent sequence is a Cauchy sequence, since*

$$\|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\|.$$

Definition 1.1.4. *A normed space X is called complete (or Banach space), if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is converges in X.*

1.1.2 Inner Product Spaces

Definition 1.1.5. *Let X be a vector space, an inner product on X is a mapping $\langle . \rangle : X \times X \longrightarrow \mathbb{K}$ such that $\forall x,\ y,\ z \in X$ and $\forall \alpha \in \mathbb{K}$ we have*

1. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
2. $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$,
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$,
4. $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Definition 1.1.6. *A Hilbert space $\mathscr{H}$ is a complete inner product space.*

1.1.3 Continuous Maps

Definition 1.1.7. *Let X and Y be vector spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $D(T)$ be a subspace of X.*

A map

$$T : D(T) \subset X \longrightarrow Y \textit{ such that : } T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

is called linear operator. If $D(T) = X$, then we write $T : X \longrightarrow Y$

Definition 1.1.8. *Let X and Y be two normed spaces, the linear operator $T : D(T) \subset X \longrightarrow Y$ is said to be bounded if there exist a real number $C > 0$ such that*

$$\|Tx\|_Y \leq C \|x\|_X, \ \forall x \in D(T). \tag{1.3}$$

Definition 1.1.9. *Let X and Y be two normed spaces where $D(T) \subset X$, and let $T : D(T) \longrightarrow Y$ be any operator (not necessarely linear), we say that T is continuous at $x_0 \in D(T)$ if*

$$\forall \varepsilon > 0, \ \exists \delta > 0 \ \textit{ such that } \|T_x - T_{x_0}\| < \varepsilon, \forall x \in D(T) \ \textit{ satisfying } \ \|x - x_0\| < \delta. \tag{1.4}$$

Definition 1.1.10. *Let X be a vector space and $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) be a scalar field, the linear operator $f : X \longrightarrow \mathbb{K}$ is caled a linear functional.*

Note that, a bounded linear functional f is a bounded linear oparator i.e

$$\exists c \in \mathbb{R} \textit{ such that } \forall x \in D(T), \ |f(x)| \leq c \|x\|.$$

Theorem 1.1.1. *[26] (**Riesz's Theorem**) If f is a bounded linear functional on a Hilbert space $\mathscr{H}$, then there exists some $y \in \mathscr{H}$ such that for every $x \in \mathscr{H}$ we have $f(x) = \langle x, y\rangle$. Where y is uniquely determined by f and has norm $\|f\| = \|y\|$.*

Definition 1.1.11. *Let T be a bounded linear operator from a Hilbert space $\mathscr{H}_1$ to another $\mathscr{H}_2$. Then there is a bounded operator $T^* : \mathscr{H}_2 \longrightarrow \mathscr{H}_1$ called the adjoint of T such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle\,,\ \forall x \in \mathscr{H}_1 \text{ and } y \in \mathscr{H}_2.$$

We say that T is self-adjoint if $T = T^*$.

Definition 1.1.12. *Let Ω be a domain in $\mathbb{R}^n$, we refer to $L^p(\Omega)$, $1 \le p < \infty$, the linear space of p^{th} order of integrable functions u on Ω and to $L^\infty(\Omega)$ as the linear space of essentially bounded functions.*

The spaces $L^p(\Omega)$, $1 \le p < \infty$, and $L^\infty(\Omega)$ are Banach spaces with respect to the norms $\|\mathrm{u}\|_{L^p} = \left(\int_\Omega |\mathrm{u}(x)|^p\, dx\right)^{\frac{1}{p}} < \infty$, and $\|\mathrm{u}\|_{L^\infty} = \underset{x\in\Omega}{esssup}\, |\mathrm{u}(x)|$, respectively. Note that for $p = 2$, the space $L^2(\Omega) = \left\{\mathrm{u} : \left(\int_\Omega |\mathrm{u}(x)|^2\, dx\right)^{\frac{1}{2}} < \infty\right\}$ is a Hilbert space with respect to the inner product $\langle \mathrm{u}, \mathrm{v}\rangle_{L^2} = \int_\Omega \mathrm{u}(x)\mathrm{v}(x)dx$.

Definition 1.1.13. *A function $\mathrm{u} : [a, b] \longrightarrow \mathbb{R}$ is called absolutely continuous (Abs.C) , if for every positive ε, there exists δ such that for any finite set of $(x_1, y_1), (x_2, y_2), \cdots, (x_k, y_k)(\subset [a, b])$ satisfies $\sum_{i=1}^{k} |y_i - x_i| < \delta$ then*

$$\sum_{i=1}^{k} |\mathrm{u}(y_i) - \mathrm{u}(x_i)| < \varepsilon.$$

1.2 Fractional Calculus

1.2.1 Special Functions

Euler Gamma Function

Definition 1.2.1. *[25] The Gamma function (or the factorial function) is defined as follows*

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \;\; \alpha > 0. \tag{1.5}$$

The integral in (1.5) is uniformly convergent for all $\alpha \in [a, b]$ where $0 < a \leq b < \infty$, so $\Gamma(\alpha)$ is a continuous function for all $\alpha > 0$. We can also define Gamma as follows

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n! n^\alpha}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n)}. \tag{1.6}$$

Now, we present some properties of the Gamma function as follows

1. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \;\; \alpha \in \mathbb{R}^+$,

2. $\Gamma(n + 1) = n(n - 1)!, \;\; n \in \mathbb{N}$,

3. $\Gamma(1) = 1$,

4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

From (2), we conclude that Gamma function is the extension of the factorial.

Beta Function

Definition 1.2.2. *[25] Also known as Euler's first integral, it is defined by two-parameter integral*

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \ \ \alpha,\beta \in \mathbb{R}^+.$$

We can express Beta in term of Gamma as follows

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.7}$$

From (1.7) it is evident that

$$B(\alpha,\beta) = B(\beta,\alpha).$$

1.2.2 Fractional Integration and Differentiation

Unlimited focus has recently been given to the theory of fractional calculus. Several forms of fractional integrals were introduced and extensively studied such as the fractional integral of Riemann-Liouville which is defined by

Definition 1.2.3. *[14] Let* $\alpha \in \mathbb{R}^+$*, the operator* $\mathscr{I}_a^\alpha$ *defined on* $L^1[a,b]$ *by*

$$\mathscr{I}_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)ds, \tag{1.8}$$

for $a \leq x \leq b$*, is called the Riemann-Liouville fractional integral operator of order* α*, with the identity operator* $\mathscr{I}_a^0 = 1$ *for* $\alpha = 0$.

In fractional integral, one of the most important operator properties is also valid as follows

Theorem 1.2.1. *[14] Let* $\alpha,\ \beta \geq 0$ *and* $f \in L^1[a,b]$, *then*

$$\mathscr{I}_a^{\alpha}\mathscr{I}_a^{\beta} f = \mathscr{I}_a^{\alpha+\beta} f,$$

holds almost everywhere on $[a,b]$. *If additionaly* $f \in C[a,b]$ *or* $\alpha+\beta \geq 1$, *the identity holds everywhere on* $[a,b]$.

Corollary 1.2.1. *[14] Under the assumptions of the previous theorem,*

$$\mathscr{I}_a^{\alpha}\mathscr{I}_a^{\beta} f = \mathscr{I}_a^{\beta}\mathscr{I}_a^{\alpha} f.$$

Example 1.2.1. *Concider the following examples*

1. *Let* $f(x) = (x-a)^{\lambda}$ *for some* $\lambda > 1$ *and* $\alpha > 0$, *then*

$$\mathscr{I}_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (s-a)^{\lambda}(x-s)^{\alpha-1} ds,$$

by the substitution $s = a + \tau(x-a)$*:*

$$\begin{aligned} \mathscr{I}_a^{\alpha} f(x) &= \tfrac{1}{\Gamma(\alpha)}(x-a)^{\alpha+\lambda} \int_0^1 \tau^{\lambda}(1-\tau)^{\alpha-1} d\tau, \\ &= \tfrac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}(x-a)^{\alpha+\lambda}. \end{aligned}$$

2. Let $f(x) = C$, such that C is arbitrary constant then

$$\begin{aligned} \mathscr{I}_a^\alpha f(x) &= \tfrac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} C ds = \tfrac{C}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} ds, \\ &= \tfrac{C}{\Gamma(\alpha+1)} (x-a)^\alpha. \end{aligned}$$

Based on the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative is defined as follows

Definition 1.2.4. *[14] Let $\alpha > 0$ and suppose m such that $m-1 < \alpha \leq m$, then the Riemann-Liouville fractional derivative of order α is defined by*

$$\begin{aligned} \mathscr{D}_a^\alpha f(x) &= \mathscr{D}^m \mathscr{I}_a^{m-\alpha} f(x), \\ &= \tfrac{1}{\Gamma(m-\alpha)} \left(\tfrac{d}{dx}\right)^m \int_a^x (x-s)^{m-\alpha-1} f(s) ds, \end{aligned}$$

for all $a \leq x \leq b$. For $\alpha = 0$, we set $\mathscr{D}_a^0 = I$ the identity operator.

Lemma 1.2.1. *[14] Let $\alpha \in \mathbb{R}^+$, and for all $m \in \mathbb{N}$ such that $m > \alpha$ we get*

$$\mathscr{D}_a^\alpha = \mathscr{D}^m \mathscr{I}_a^{m-\alpha}.$$

Theorem 1.2.2. *[14] Suppose that $\alpha_1, \alpha_2 \geq 0$, and let $\psi \in L^1[a,b]$ and $f = \mathscr{I}^{\alpha_1+\alpha_2}\psi$ then*

$$\mathscr{D}_a^{\alpha_1} \mathscr{D}_a^{\alpha_2} f = \mathscr{D}_a^{\alpha_1+\alpha_2} f.$$

Example 1.2.2. *Let $f(x) = (x-a)^\lambda$, for $\lambda > -1$ and let $\alpha > 0$, then*

$$\mathscr{D}_a^\alpha f(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} (x-a)^{\lambda-\alpha}.$$

To prove this, let $m \in \mathbb{N}$ such that $m = \lceil \alpha \rceil$, then

$$\begin{aligned}
\mathscr{D}_a^{\alpha} f(x) &= \mathscr{D}_a^{\alpha} (x-a)^{\lambda}, \\
&= \mathscr{D}^{m} \mathscr{J}_a^{m-\alpha} (x-a)^{\lambda}, \\
&= \tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} (x-a)^{\lambda-\alpha}.
\end{aligned}$$

Remark 1.2.1. *The property $\mathscr{D}_a^{\alpha} \mathscr{D}_a^{\beta} f = \mathscr{D}_a^{\alpha+\beta} f = \mathscr{D}_a^{\beta} \mathscr{D}_a^{\alpha} f$ is not satisfies in both equalities.*

Theorem 1.2.3. *[14] Let $\alpha \geq 0$, then for all $f \in L^1[a,b]$ we have*

$$\mathscr{D}_a^{\alpha} \mathscr{I}_a^{\alpha} f = f.$$

Almost everywhere.

Concerning the relation between Riemann-Liouville integral and derivative, we present the following theorem.

Theorem 1.2.4. *[14] Let $\alpha > 0$, if there is a function $\psi \in L^1[a,b]$ such that $f = \mathscr{I}_a^{\alpha} \psi$ then*

$$\mathscr{I}_a^{\alpha} \mathscr{D}_a^{\alpha} f = f.$$

Almost everywhere.

Theorem 1.2.5. *[14] Let $\alpha > 0$ and $m-1 < \alpha \leq m$, suppose f such that $\mathscr{I}_a^{m-\alpha} f \in \mathcal{A}^m[a,b]$ then*

$$\mathscr{I}_a^{\alpha} \mathscr{D}_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{q \to a^+} \mathscr{D}^{m-k-1} \mathscr{I}_a^{m-\alpha} f(q).$$

In particular, if $0 < \alpha < 1$ *we get*

$$\mathscr{I}_a^{\alpha}\mathcal{D}_a^{\alpha}f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{q\to a^+} \mathscr{I}_a^{1-\alpha}f(q).$$

M. Caputo introduced a new interpretation of a fractional derivative in 1967, which is referred to as the Caputo fractional derivative. The most significant aspect of his work was demonstrating its correlation with fractional Riemann-Liouville differential and integral operators.

Definition 1.2.5. *[14] Let* $\alpha \geq 0$*, and* $m-1 < \alpha \leq m$*, then we define the Caputo fractional differential operator* $\mathscr{D}_{*a}^{\alpha}$ *of order* α *as follows*

$$\begin{aligned} \mathscr{D}_{*a}^{\alpha}f &= \mathscr{I}_a^{m-\alpha}\mathscr{D}^m f \\ &= \tfrac{1}{\Gamma(m-\alpha)}\int_a^x (x-s)^{m-\alpha-1}(\tfrac{d}{ds})^m f(s)ds, \end{aligned}$$

for all $a \leq x \leq b$*.*

Example 1.2.3. *Let* $f(x) = (x-a)^{\lambda}$ *for some* $\lambda \geq 0$*, then*

$$\mathscr{D}_{*a}^{\alpha}(x-a)^{\lambda} = \mathscr{I}_a^{m-\alpha}\mathscr{D}^m(x-a)^{\lambda}, \ \ \alpha > 0.$$

Then,

$$\mathscr{D}_{*a}^{\alpha}(x-a)^{\lambda} = \begin{cases} 0, & if\ \lambda \in \{0, 1, \cdots, m-1\}, \\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}(x-a)^{\lambda-\alpha}, & if\ \lambda \in \mathbb{N}\ and\ \lambda \geq m\ or\ \lambda \notin \mathbb{N}\ and\ \lambda > m-1. \end{cases}$$

We notice that the Caputo derivative is also a left inverse of the Riemann-Liouville integral operator, considering the interaction of Riemann-Liouville integral and Caputo differential operators.

Theorem 1.2.6. *[14] Let* $\alpha \geq 0$ *and* f *be a continuous function, then*

$$\mathscr{D}_{*a}^{\alpha} \mathscr{I}_a^{\alpha} f = f.$$

Moreover, the Caputo derivative is not the right inverse of the Riemann-Liouville integral.

Theorem 1.2.7. *[14] Let* $\alpha \geq 0$*, such that* $m-1 < \alpha \leq m$*, and* $f \in \mathcal{A}^m[a,b]$*, then*

$$\mathscr{I}_a^{\alpha} \mathscr{D}_{*a}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{\mathscr{D}^k f(a)}{k!}(x-a)^k.$$

Chapter 2 Reproducing Kernel Hilbert Spaces

A reproducing kernel Hilbert space, which is constructed from a Hilbert space $\mathscr{H}$, mandates that all Dirac evaluation functionals in $\mathscr{H}$ are both continuous and bounded.

A Dirac functional at an element $x \in X$ is a functional $\delta_x \in \mathscr{H}$ such that $\delta_x(f) = f(x)$. Note that δ_x is bounded if $\exists \mathcal{M} > 0$ *such that* $\|\delta_x f\|_{\mathbb{R}} \leq \mathcal{M} \|f\|_{\mathscr{H}}, \ \forall f \in \mathscr{H}$. The significance of this is due to Riesz's Theorem (1.1.1). When we convert this theorem into Dirac evaluation functionals, we obtain that for every δ_x, there is a unique vector k_x in $\mathscr{H}$ such that $\delta_x(f) = f(x) = \langle f, \mathrm{k}_x \rangle_{\mathscr{H}}$.

2.1 Reproducing Kernel Hilbert Space

Definition 2.1.1. *[3] Consider $\mathscr{H}$ to be a Hilbert space of functions $f : X \longrightarrow \mathbb{K}$ on a set X. A function $\mathrm{k} : X \times X \longrightarrow \mathbb{C}$ is a reproducting kernel of $\mathscr{H}$ if the following properties are satisfied*

1. $\mathrm{k}(., x) \in \mathscr{H}$, *For every* $x \in X$,
2. $\langle f, \mathrm{k}(., x)\rangle_{\mathscr{H}} = f(x)$, *For every* $f \in \mathscr{H}$ *and* $x \in X$, *(Reproducing Property).*

The second condition implies that the function f evaluated at x is reproduced by the inner product of f with k. *Additionally, the first condition can be rewritten as: for all $x \in X$, $\mathrm{k}_x(y) = \mathrm{k}(x, y) \in \mathscr{H}$, $\forall y \in X$. Thus, utilizing the reproducing property to the function k_x at y, we get:*

$$\mathrm{k}_x(y) = \langle \mathrm{k}_x, \mathrm{k}_y \rangle \,, \ \forall x, y \in X.$$

therefore, $\forall x \in X$, we obtain $\|\mathrm{k}_x\|^2 = \langle \mathrm{k}_x, \mathrm{k}_x \rangle = \mathrm{k}(x, x)$.

If there is a reproducing kernel k *of a Hilbert functions space $\mathscr{H}$, then $\mathscr{H}$ is called a reproducing kernel Hilbert space (RKHS). We denote the RKHS by $\mathscr{H}_{\mathrm{k}}$, and $\langle ., . \rangle_{\mathscr{H}_{\mathrm{k}}}$, $\|.\|_{\mathscr{H}_{\mathrm{k}}}$ represent the inner product and the norm respectively.*

Theorem 2.1.1. *[3] If a Hilbert space $\mathscr{H}$ of functions defined on a set X has a reproducing kernel, then the reproducing kernel $\mathrm{k}(x, y)$ is uniquely determined by the Hilbert space $\mathscr{H}$.*

Theorem 2.1.2. *[3] Let $\mathscr{H}$ be a Hilbert functions space on X, then there is a reproducing kernel* k *of $\mathscr{H}$ if and only if for every $x \in X$, the Dirac functional $\delta_x : f \longrightarrow f(x)$ is a bounded linear functional on $\mathscr{H}$.*

Definition 2.1.2. *[3] Let $\mathrm{k} : X \times X \longrightarrow \mathbb{C}$ be a complex-valued fucntion on a set X, then*

1. k *is Hermitian if for every finite set of points $y_1, ..., y_n \subseteq X$, and any complex numbers $c_1, ..., c_2$, we have*

$$\sum_{i,j=1}^{\infty} \overline{c_i} c_j \ \mathrm{k}(y_i, y_j) \ \in \mathbb{R}$$

2. k *is positive definite if*

$$\sum_{i,j=1}^{\infty} \overline{c_i} c_j \ \mathrm{k}(y_i, y_j) \quad \geq 0$$

Theorem 2.1.3. *[3] The reproducing kernel* $\mathrm{k}(x, y)$ *of a reproducing kernel Hilbert space* $\mathscr{H}$ *is a positive definite kernel.*

Proof : We have

$$\begin{aligned} 0 &\leq \|\textstyle\sum_{i=1}^{n} c_i \mathrm{k}_{x_i}\|^2 = \langle \textstyle\sum_{i=1}^{n} c_i \mathrm{k}_{x_i} \ , \ \textstyle\sum_{i=1}^{n} c_i \mathrm{k}_{x_i} \rangle \, , \\ &= \textstyle\sum_{i=1}^{n} \textstyle\sum_{j=1}^{n} c_i \overline{c_j} \left\langle \mathrm{k}_{x_i} \ , \ \mathrm{k}_{x_j} \right\rangle , \\ &= \textstyle\sum_{i=1}^{n} \textstyle\sum_{j=1}^{n} c_i \overline{c_j} \mathrm{k}(x_i, x_j). \end{aligned}$$

Hence, $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \mathrm{k}(x_i, x_j) \geq 0$. □

Remark 2.1.1. *Let* $\mathscr{H}$ *be a RKHS, and* $\mathrm{k}(x, y)$ *its kernel on* X*. Then for every* $x, y \in X$*, we have the following*

1. $|\mathrm{k}(x, y)|^2 \leq \mathrm{k}(x, x)\mathrm{k}(y, y)$*,*

2. *for* $x^* \in X$*, then the following are equivalent*

   - $\mathrm{k}(x^*, x^*) = 0$*,*
   - $\mathrm{k}(x^*, y) = 0, \ \forall y \in X$*,*
   - $f(x^*) = 0, \ \forall f \in \mathscr{H}$*.*

We can demonstrate (1) by applying the Schwarz inequality in $\mathscr{H}$*, so we obtain*

$$|\mathrm{k}(x, y)|^2 = |\langle \mathrm{k}_x, \mathrm{k}_y \rangle|^2 \leq \|\mathrm{k}_x\|^2 \|\mathrm{k}_y\|^2 = \langle \mathrm{k}_x, \mathrm{k}_x \rangle \langle \mathrm{k}_y \mathrm{k}_y \rangle = \mathrm{k}(x, x)\mathrm{k}(y, y).$$

For (2) it follows by (1) that

$$|\mathrm{k}(x^*, y)|^2 \leq \mathrm{k}(x^*, x^*)\mathrm{k}(y, y) = 0.$$

Hence, $\mathrm{k}(x^*, x^*) = 0$ *is equivalent with* $\mathrm{k}(x^*, y) = 0, \forall y \in X$. *Moreover, by the reproducing property* $\mathrm{k}(x^*, y) = 0$, $\forall y \in X$ *if and only if* $f(x^*) = 0$ *for every* $f \in \mathscr{H}$.

Theorem 2.1.4. *[3] Every sequence of functions* $(g_n)_{n\geq 1}$ *that strongly converge to a function* g *in* $\mathscr{H}_{\mathrm{k}}(x)$*, it also converges in the pointwise sense, which means* $\lim_{n\to\infty} g_n(x) = g(x)$, $\forall x \in X$. *Furthermore, this convergence is uniform on every subset of* X *on which* $x \longrightarrow \mathrm{k}(x, x)$ *is bounded.*

Proof : For $x \in X$, using the Schwwarz inequality and the reproducing property, we get:

$$\begin{aligned} |g(x) - g_n(x)| &= |\langle g\ ,\ \mathrm{k}_x\rangle - \langle g_n\ ,\ \mathrm{k}_x\rangle|\,, \\ &= |\langle g - g_n\ ,\ \mathrm{k}_x\rangle|\,, \\ &\leq \|g - g_n\|\ \|\mathrm{k}_x\|\,, \\ &= \|g - g_n\|\ \mathrm{k}(x, x)^{\frac{1}{2}}. \end{aligned}$$

Hence, $\lim_{n\to\infty} g_n(x) = g(x)$, $\forall x \in X$. Moreover, from the above inequality it is clear that this converge is uniform on every subset of X on $x \longrightarrow \mathrm{k}(x, x)$ is bounded. □

Definition 2.1.3. *Consider the non-negative integer* m*, and let* $u \in L^2[a, b]$*, then the function space* $\mathscr{W}_2^m[a, b]$ *is defined as follows*

$$\mathscr{W}_2^m[a, b] = \{u | u^{(i)} \ \textit{is Abs.C},\ i = 1, ..., m - 1,\ \textit{and}\ u^{(m)} \in L^2[a, b]\}.$$

The inner product and the norm are defined respectively in the function space $\mathscr{W}_2^m[a, b]$

as follows:

$$\langle u,v\rangle_{\mathscr{W}_2^m[a,b]} := \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \tag{2.1}$$

and

$$\|u\|_{\mathscr{W}_2^m[a,b]} = (\langle u,u\rangle_{\mathscr{W}_2^m[a,b]})^{\frac{1}{2}}, \tag{2.2}$$

for all functions $u(x), v(x)$ *in* $\mathscr{W}_2^m[a,b]$.

Theorem 2.1.5. *[7] The function space* $\mathscr{W}_2^m[a,b]$ *is a Hilbert space.*

Theorem 2.1.6. *[3] The function space* $\mathscr{W}_m^2[a,b]$ *is a reproducing kernel space. That is,* $\forall x \in [a,b]$, $\forall u(y) \in \mathscr{W}_2^m[a,b]$, $\exists\ \mathrm{k}_x(y) \in \mathscr{W}_2^m[a,b]$, $y \in [a,b]$ *such that* $\langle u(y), \mathrm{k}_x(y)\rangle = u(x)$*, and* $\mathrm{k}_x(y)$ *is called the reproducing kernel function of the space* $\mathscr{W}_2^m[a,b]$.

2.2 Reproducing Kernel Function

This section provides the various representations of the reproducing kernel functions in the space $\mathscr{W}_2^m[a,b]$. These representations are presented as piecewise polynomials with a degree of $2m-1$. additionally, we will present some remarks and corollaries related to these kernel functions. Several examples of such kernel functions are given in space $\mathscr{W}_2^1[a,b]$ at the end of this section.

Let's now determine the expression form of the reproduction kernel function $\mathrm{k}_x(y)$ in the space $\mathscr{W}_2^m[a,b]$. Suppose that $\mathrm{k}_x(y)$ is the reproducing kernel function of the space $\mathscr{W}_2^m[a,b]$, so for all fixed $x \in [a,b]$ and every $u(y) \in \mathscr{W}_2^m[a,b]$, $y \in [a,b]$ we have $\langle u(y), \mathrm{k}_x(y)\rangle = u(x)$, through the equations (2.1) and (2.2) we get

$$\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)\mathrm{k}_x^{(i)}(a) + \int_a^b u^{(m)}(y)\mathrm{k}_x^{(m)}(y)dy, \tag{2.3}$$

using the integration by part for the right-hand of equation (2.3) we obtain

$$\int_a^b u^{(m)}(y)\mathrm{k}_x^{(m)}(y)dy = \sum_{i=0}^{m-1}(-1)^i u^{(m-i-1)}(y)\mathrm{k}_x^{(m+i)}(y)|_{y=a}^b + \int_a^b (-1)^m u(y)\mathrm{k}_x^{(2m)}(y)dy.$$

Assume that $j = m-i-1$, then the first term from the right side of the above formula can be rewritten as follows

$$\sum_{i=0}^{m-1}(-1)^i u^{(m-i-1)}(y)\mathrm{k}_x^{(m+i)}(y)|_{y=a}^b = \sum_{j=0}^{m-1}(-1)^{m-j-1}u^{(j)}(y)\mathrm{k}_x^{(2m-j-1)}(y)|_{y=a}^b.$$

After a certain simplification, equation (2.3) becomes

$$\begin{aligned}\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} &= \sum_{i=0}^{m-1} u^{(i)}(a)(\mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a)) \\ &+ \sum_{i=0}^{m-1}(-1)^{m-i-1}u^{(i)}(b)\mathrm{k}_x^{(2m-i-1)}(b) + \int_a^b (-1)^m u(y)\mathrm{k}_x^{(2m)}(y)dy.\end{aligned}$$

Since $\mathrm{k}_x(y), u(y) \in \mathscr{W}_2^m[a,b]$, it implies that

$$\mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a) = 0,\ \mathrm{k}_x^{(2m-i-1)}(b) = 0,\ i = 0, \cdots, m-1.$$

Then $\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} = \int_a^b u(y)((-1)^m \mathrm{k}_x^{(2m)}(y))dy$.

Now, let δ the dirac-delta function, for all $x \in [a,b]$, if $(-1)^{(m)}\mathrm{k}_x^{(2m)}(y) = \delta(x-y)$, then

$$\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} = \int_a^b u(y)\delta(x-y)dy = u(x),$$

clearly, $\mathrm{k}_x(y)$ is the reproducing kernel of the space $\mathscr{W}_2^m[a,b]$, Hence $\mathrm{k}_x(y)$ is solution of the following generalized differential equations

$$\begin{cases} (-1)^m \mathrm{k}_x^{(2m)}(y) &= \delta(x-y), \\ \mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a) &= 0,\ i=0,\cdots,m-1\ , \\ \mathrm{k}_x^{(2m-i-1)}(b) &= 0,\ i=0,\cdots,m-1. \end{cases} \tag{2.4}$$

When $x \neq y$

$$(-1)^m \mathrm{k}_x^{(2m)}(y) = 0, \tag{2.5}$$

with the boundary conditions

$$\mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a) = 0,\ \mathrm{k}_x^{(2m-i-1)}(b) = 0,\ i=1,\cdots,m-1. \tag{2.6}$$

For the equations (2.5), $\lambda^{2m} = 0$ is the characteristic equation, and $\lambda = 0$ is their characteristic values with $2m$ multiple roots, so the general solution of equation (2.5) is as follows

$$\mathrm{k}_x(y) = \begin{cases} \sum_{i=0}^{2m-1} \mathcal{P}_i(x) y^i, & y \le x, \\ \sum_{i=0}^{2m-1} \mathcal{Q}_i y^i, & y > x. \end{cases} \tag{2.7}$$

Moreover, since $(-1)^m \mathrm{k}_x^{2m}(y) = \delta(x-y)$, we have

$$\mathrm{k}_x^{(i)}(x+0) = \mathrm{k}_x^{(i)}(x-0),\ i=0,\cdots,2m-2, \tag{2.8}$$

by the integration of $(-1)^m \mathrm{k}_x^{(2m)}(y) = \delta(x-y)$ from $x-\xi$ to $x+\xi$ with respect to

y and let $\xi \longrightarrow 0$, we have the jump degree of $\mathrm{k}_x^{(2m-1)}(y)$ at $y = x$ given by

$$(-1)^m(\mathrm{k}_x^{(2m-1)}(x+0) - \mathrm{k}_x^{(2m-1)}(x-0)) = 1. \tag{2.9}$$

We have $2m$ equations: equation (2.8) and (2.9) provided $2m$ conditions for determining the coefficients $\mathcal{P}_i(x)$ and $\mathcal{Q}_i(x)$ in (2.8), for $i = 0, 1, \cdots, 2m-1$. Further, equation (2.6) provided $2m$ boundary conditions. Thus, we obtain a total of $2m$ equations. It is evident that these $4m$ equations are linear equations with the variables $\mathcal{P}_i(x)$ and $\mathcal{Q}_i(x)$, and the unknown coefficients $\mathcal{P}_i(x)$ and $\mathcal{Q}_i(x)$ of equation (2.7) can be computed by using Mathematica 11.0 software package.

Some important properties of the reproducing kernel $\mathrm{k}_x(y)$ are provided by the following corollary.

Corollary 2.2.1. *Let* $\mathrm{k}_x(y)$ *be the reproducing kernel of the space* $\mathscr{W}_2^m[a, b]$*, then* $\mathrm{k}_x(y)$ *is symmetric, unique and* $\mathrm{k}_x(y) \geq 0$ *, for all fixed* $x \in [a, b]$*.*

Proof : By the reproducing property, we have

$$\mathrm{k}_x(y) = \langle \mathrm{k}_x(.), \mathrm{k}_y(.) \rangle = \langle \mathrm{k}_y(.), \mathrm{k}_x(.) \rangle = \mathrm{k}_y(x).$$

Now, assume that $\mathrm{k}_x(y)$ and $\widetilde{\mathrm{k}}_x(y)$ be all the reproducing kernel of the space $\mathscr{W}_2^m[a, b]$, then

$$\mathrm{k}_x(y) = \langle \mathrm{k}_x(.), \widetilde{\mathrm{k}}_y(.) \rangle = \langle \widetilde{\mathrm{k}}_y(.), \mathrm{k}_x(.) \rangle = \widetilde{\mathrm{k}}_y(x).$$

since $\widetilde{\mathrm{k}}_x(y)$ is symmetric, we get the unique representation of $\mathrm{k}_x(y)$. For the last property, we note that

$$\mathrm{k}_x(x) = \langle \mathrm{k}_x(.), \mathrm{k}_x(.)\rangle = \|\mathrm{k}_x(.)\|^2 \geq 0.$$

□

Now we provide some expressions of reproducing kernel function in the space $\mathscr{W}_2^1[a,b]$ with respect to various norms by using the the method suggested in this section.

Example 2.2.1. *Let the space* $\mathscr{W}_2^1[a,b] = \{u : [a,b] \longrightarrow \mathbb{R} :\ u(x)$ *is Abc.C and* $u'(x) \in L^2[a,b]\}$*, the inner product and the norm in this space are given by*

$$\langle u, v\rangle_{\mathscr{W}_2^1} = u(a)v(a) + \int_a^b u'(y)v'(y)dy, \text{ and } \|u\|_{\mathscr{W}_2^1} = \langle u, u\rangle^{\frac{1}{2}},\ \forall u(x), v(x) \in \mathscr{W}_2^1[a,b].$$

We apply the integration by parts to find the reproducing kernel function $\mathrm{k}_x(y)$

$$\langle u, \mathrm{k}_x\rangle_{\mathscr{W}_2^1} = u(a)\mathrm{k}_x(a) + u(y)\mathrm{k}'_x(y)|_{y=a}^{b} - \int_a^b u(y)\mathrm{k}''_x(y)dy.$$

Since $u(y), \mathrm{k}_x(y) \in \mathscr{W}_2^1[a,b]$*, we get* $\mathrm{k}_x(a) - \mathrm{k}'_x(a) = 0$ *and* $\mathrm{k}'_x(b) = 0$*. So, we must solve the BVP.*

$$\begin{cases} -\mathrm{k}''_x(y) & = & \delta(x-y), \\ \mathrm{k}_x(a) - \mathrm{k}'_x(a) & = & 0, \\ \mathrm{k}'_x(b) & = & 0. \end{cases}$$

$\lambda^2 = 0$ *is the characteristic equation of the differential equation* $-\mathrm{k}''_x(y) = 0$*, and the* *characteristic value is* $\lambda = 0$ *with 2 multiple roots. Then*

$$\mathrm{k}_x(y) = \begin{cases} \mathcal{P}_1(x) + \mathcal{P}_2(x)y, & y \le x, \\ \mathcal{Q}_1(x) + \mathcal{Q}_2(x)y, & y > x. \end{cases}$$

Moreover, by using the equations (2.8) and (2.9) we get $\mathrm{k}_x(x+0) = \mathrm{k}_x(x-0)$ *and* $\mathrm{k}'_x(x+0) - \mathrm{k}'_x(x-0) = 1$*, Accordingly, the unknown coefficients* $\mathcal{P}_i(x)$*, and* $\mathcal{Q}_i(x)$*,* $i = 1, 2$ *can be found by resolving the following equations*

1) $\mathrm{k}_x(a) - \mathrm{k}'_x(a) = 0,$

2) $\mathrm{k}'_x(b) = 0,$

3) $\mathrm{k}_x(x+0) = \mathrm{k}_x(x-0),$

4) $\mathrm{k}'_x(x+0) - \mathrm{k}'_x(x-0) = -1.$

Thus, the reproducing kernel function is given by

$$\mathrm{k}_x(y) = \begin{cases} y - a + 1, & y \le x, \\ x - a + 1, & y > x. \end{cases}$$

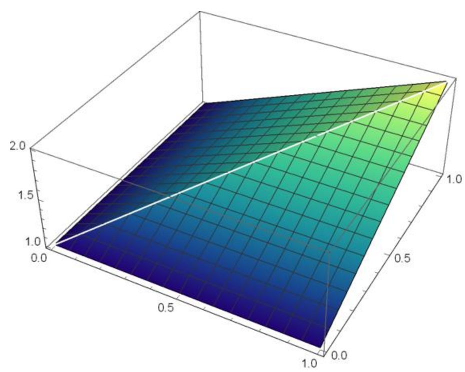

Figure 2.1: Image of the reproducing kernel function $\mathrm{k}_x(y)$ of the space $\mathscr{W}_2^1[0,1]$

Example 2.2.2. *Consider the space* $\mathscr{W}_2^1[a,b] = \{u :\ u(x)$ *is Abs.C,* $u'(x) \in$

$L^2[a,b]$ and $u(a) = u(b) = 0\}$. The inner product and the norm in this space are given respectively by

$$\begin{cases} \langle u, v\rangle_{\mathscr{W}_2^1} = \int_a^b u'(y)v'(y)dy, \\ \|u\| = (\langle u, u\rangle)^{\frac{1}{2}}, \end{cases} \quad \text{where } u(x),\ v(x) \in \mathscr{W}_2^1. \tag{2.10}$$

Similarly, as in Example (2.2.1) we have

$$\mathrm{R}_x(y) = \begin{cases} c_1(x) + c_2(x)y, \ y \le x \ , \\ d_1(x) + d_2(x)y, \ y > x \ . \end{cases}$$

The unknown coefficients $c_i(x)$, and $d_i(x)$, $i = 1, 2$ can be found by resolving the following equations

1) $\mathrm{R}_x(a) = 0,$

2) $\mathrm{R}_x(b) = 0,$

3) $\mathrm{R}_x(x+0) = \mathrm{R}_x(x-0),$

4) $\mathrm{R}'_x(x+0) - \mathrm{R}'_x(x-0) = -1.$

Then, the reproducing kernel $\mathrm{R}_x(y)$ is given by

$$\mathrm{R}_x(y) = \begin{cases} \dfrac{(b-x)(a-y)}{a-b}, \ y \le x, \\ \dfrac{(a-x)(b-y)}{a-b}, \ y > x. \end{cases}$$

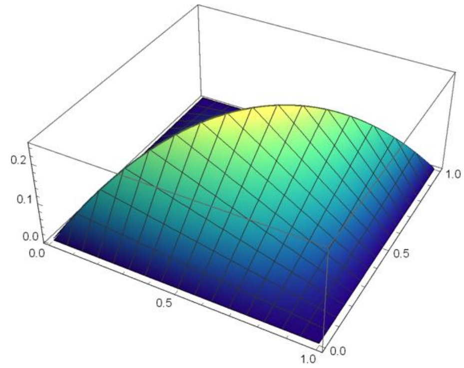

Figure 2.2: Image of the reproducing kernel function $\mathrm{R}_x(y)$ of the space $\mathscr{W}_2^1[0,1]$

Example 2.2.3. *Consider the space $\mathscr{W}_2^1[a,b]$ defined as the same set of functions in Example(2.2.1), and specify a new inner product in the space $\mathscr{W}_2^1[a,b]$ by $\langle u,v\rangle_{\mathscr{W}_2^1} = \int_a^b (u(y)v(y)+u'(y)v'(y))dy$, such that $u(x),v(x)\in\mathscr{W}_2^1[a,b]$, and with the same norm. Via integration by parts of $\langle u,\mathrm{P}_x\rangle_{\mathscr{W}_2^1} = \int_a^b (u(y)\mathrm{P}_x(y)+u'(y)\mathrm{P}_x'(y))dy$, we get*

$$\langle u,\mathrm{P}_x\rangle_{\mathscr{W}_2^1} = u(y)\mathrm{P}_x'(y)|_{y=a}^{b} + \int_a^b u(y)(\mathrm{P}_x(y)-\mathrm{P}_x''(y))dy.$$

Since $u(y),\mathrm{P}_x(y)\in\mathscr{W}_2^1[a,b]$, we get $\mathrm{P}_x'(a)=\mathrm{P}_x'(b)=0$. Then $\mathrm{P}_x(y)-\mathrm{P}_x''(y) = \delta(x-y)$. The characteristic equation is $1-\lambda^2=0$, and the characteristic values are $\lambda=-1,1$. So

$$\mathrm{P}_x(y) = \begin{cases} a_1(x)e^{-y}+a_2(x)e^{y}, & y\le x, \\ b_1(x)e^{-y}+b_2(x)e^{y}, & y>x. \end{cases}$$

By solving the following equations

$$1)\ \mathrm{P}'_x(a) = 0,$$

$$2)\ \mathrm{P}'_x(b) = 0,$$

$$3)\ \mathrm{P}_x(x+0) = \mathrm{P}_x(x-0),$$

$$4)\ \mathrm{P}'_x(x+0) - \mathrm{P}'_x(x-0) = -1,$$

we get the unknown coefficients $a_i(x),\ b_i(x),\ i = 1, 2,$ *and so the reproducing kernel function is given by*

$$\mathrm{P}_x(y) = \begin{cases} -\dfrac{e^{-(x+y)}(e^{2b}+e^{2x})(e^{2a}+e^{2y})}{2(e^{2a}-e^{2b})}, & y \leq x, \\ -\dfrac{e^{-(x+y)}(e^{2a}+e^{2x})(e^{2b}+e^{2y})}{2(e^{2a}-e^{2b})}, & y > x\ . \end{cases}$$

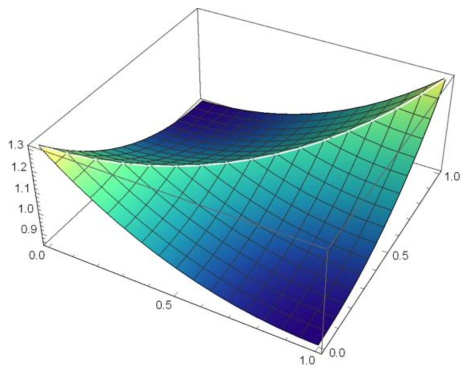

Figure 2.3: Image of the reproducing kernel function $\mathrm{P}_x(y)$ of the space $\mathscr{W}_2^1[0,1]$

2.3 The Reproducing Kernel Method

In this section, we will describe an iterative approach for constructing and determining the solution of the general m^{th} order BVP. We considering the general form of the BVP as follows

$$u^{(m)}(x)+a_1(x)u^{(m-1)}(x)+\cdots+a_{m-1}(x)u'(x)=\mathcal{F}(x,u(x)),\ \ a\le x\le b, \qquad (2.11)$$

subject to the BCs

$$\begin{cases} u^{(i)}(a)=e_i, & i=0,1,\cdots,r-1,\\ u^{(i)}(b)=d_i, & i=r,\cdots,m-1. \end{cases} \qquad (2.12)$$

Where $a_i(x),\ i=1,\cdots,m-1$, are continuous real-valued functions, $e_i,\ 0\le i\le r-1,\ and\ d_i,\ r\le i\le m-1$ are real constants, and $u(x)$ denote the unknown function to be determined, and $\mathcal{F}$ is a linear or nonlinear continuous function depending on the problem discussed.

For solving of BVP (2.11) and (2.12), we use the RKHS method. First, we create a reproducing kernel space $\mathscr{W}_2^m[a,b]$ such that every function satisfies the homogeneous BCs (2.12), and then we use the space $\mathscr{W}_2^1[a,b]$. The inner product and the norm in the space $\mathscr{W}_2^m[a,b]$ are obtained as in equations (2.1), and (2.2) respectively.

Let $\mathrm{R}_x(y)[a,b],\ \mathrm{k}_x(y)[a,b]$ be the reproducing kernel functions of the spaces $\mathscr{W}_2^1[a,b]$ and $\mathscr{W}_2^{m+1}[a,b]$ respectively. We define a differential operator $\mathcal{L}:\mathscr{W}_2^{m+1}[a,b]\to\mathscr{W}_2^1[a,b]$, such that

$$\mathcal{L}u(x)=u^{(m)}(x)+a_1(x)u^{(m-1)}(x)+\cdots+a_{m-1}(x)u'(x),$$

Through the following lemma, we can demonstrate that $\mathcal{L}$ is a bounded operator.

Lemma 2.3.1. *If* $u(x) \in \mathscr{W}_2^{m+1}[a,b]$, *then*

$$|u^{(i)}(x)| \leq \mathcal{M}_i \|u(x)\|_{\mathscr{W}_2^{m+1}},$$

such that $\mathcal{M}_i$ *are constants for* $i = 0, 1, \cdots, m$.

Proof : By the reproducing property of $\mathrm{k}_x(y)$ and Schwarz inequality, and since $\mathrm{k}_x^{(i)}(y)$ are uniformly bounded about x and y for $i = 0, 1, \cdots, m$, then

$$\begin{aligned} |u(x)| = |\langle u(y), \mathrm{k}_x(y) \rangle_{\mathscr{W}_2^{m+1}}| &\leq \|\mathrm{k}_x(y)\|_{\mathscr{W}_2^{m+1}} \|u(y)\|_{\mathscr{W}_2^{m+1}}, \\ &\leq \mathcal{M}_0 \|u(x)\|_{\mathscr{W}_2^{m+1}}. \end{aligned}$$

Moreover, from the representation of $\mathrm{k}_x(y)$, we can obtain,

$$\begin{aligned} |u^{(i)}(x)| = |\langle u(y), \mathrm{k}_x^{(i)}(y) \rangle_{\mathscr{W}_2^{m+1}}| &\leq \|\mathrm{k}_x^{(i)}(y)\|_{\mathscr{W}_2^{m+1}} \|u(y)\|_{\mathscr{W}_2^{m+1}} \\ &\leq \mathcal{M}_i \|u(x)\|_{\mathscr{W}_2^{m+1}}. \end{aligned}$$

□

Thus, after homogenizing the BC's (2.12), the BVPs (2.11), (2.12) can be transformed to the equivalent form as follows

$$\mathcal{L}u(x) = \mathcal{F}(x, u(x)), \; a \leq x \leq b, \tag{2.13}$$

$$u^{(i)}(a) = 0, \; i = 0, 1, \cdots, r-1, \; u^{(i)}(b) = 0, \; i = r, r+1 \cdots, m-1, \tag{2.14}$$

such that $u(x) \in \mathscr{W}_2^{m+1}[a,b]$ and $\mathcal{F}(x, u(x)) \in \mathscr{W}_2^1[a,b]$.

Now, we create an orthogonal function system for the space $\mathscr{W}_2^{m+1}[a,b]$, we choose a countable dense set $\{x_i\}_{i=1}^{\infty}$ of $[a,b]$, and let $\Phi_i(x) = \mathrm{R}_{x_i}(y)$. Thus by the properties of $\mathrm{R}_{x_i}(y)$, for all $u(x) \in \mathscr{W}_2^1[a,b]$, it implies that

$$\langle u(x), \Phi_i(x)\rangle_{\mathscr{W}_2^1} = \langle u(x),\ \mathrm{R}_{x_i}(y)\rangle_{\mathscr{W}_2^1} = u(x_i).$$

In addition, let $\mathcal{L}^*$ be the adjoint operator of $\mathcal{L}$ such that $\Psi_i(x) = \mathcal{L}^*\Phi_i(x)$. Clearly, $\Psi_i(x) \in \mathscr{W}_2^{m+1}[a,b]$. With respect to the properties of $\mathrm{k}_x(y)$, $\forall\ i = 1, 2, ...$, we have

$$\begin{aligned}\langle u(x), \Psi_i(x)\rangle_{\mathscr{W}_2^{m+1}} &= \langle u(x), \mathcal{L}^*\Phi_i(x)\rangle_{\mathscr{W}_2^{m+1}},\\ &= \langle \mathcal{L}u(x), \Phi_i(x)\rangle_{\mathscr{W}_2^1},\\ &= \mathcal{L}u(x_i).\end{aligned}$$

Lemma 2.3.2. $\Psi_i(x)$ *can be written on the following form*

$$\Psi_i(x) = \mathcal{L}_y\ \mathrm{k}_x(y)|_{y=x_i}.$$

Proof : It is clear that

$$\begin{aligned}\Psi_i(x) = \mathcal{L}^*\Phi_i(x) &= \langle \mathcal{L}^*\Phi_i(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^{m+1}},\\ &= \langle \Phi_i(y),\ \mathcal{L}\ \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^1},\\ &= \mathcal{L}_y\ \mathrm{k}_x(y)|_{y=x_i}.\end{aligned}$$

□

Theorem 2.3.1. *Assume that the operator* $\mathcal{L}$ *is invertible, and if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[a,b]$*, then* $\{\Psi_i\}_{i=1}^{\infty}$*, is the complete function system of the space* $\mathscr{W}_2^{m+1}[a,b]$*.*

Proof : For every fixed $u(x) \in \mathscr{W}_2^{m+1}[a,b]$, let $\langle u(x), \Psi_i^1(x)\rangle = 0,\ \forall i = 1, 2, \cdots,$

that is

$$\begin{aligned}\langle u(x), \Psi_i(x)\rangle_{\mathscr{W}_2^{m+1}} &= \langle u(x), \mathcal{L}^*\Phi_i(x)\rangle_{\mathscr{W}_2^{m+1}},\\ &= \langle \mathcal{L}u(x), \Phi_i(x)\rangle_{\mathscr{W}_2^1},\\ &= \mathcal{L}u(x_i) = 0.\end{aligned}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[a, b]$, then $\mathcal{L}u(x) = 0$, it follows that $u(x) = 0$ because $\mathcal{L}^{-1}$ exist and $u(x)$ is continuous. □

Now, the orthonormal function system $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ of the space $\mathscr{W}_2^{m+1}[a, b]$, it can be derived from Gram-Schmidt orthogonalization process of $\{\Psi_i(x)\}_{i=1}^{\infty}$ as follows

$$\overline{\Psi}_i(x) = \sum_{k=1}^{i} B_{ik}\Psi_k(x), i = 1, 2... \tag{2.15}$$

where B_{ik} are positive orthogonalization coefficients and are given by

$$B_{11} = \frac{1}{\|\Psi_1\|},\ B_{ii} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} C_{ik}^2}},\ B_{ij} = \frac{-\sum_{k=1}^{i-1} C_{ik}B_{kj}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} C_{ik}^2}}\ j < i, \tag{2.16}$$

such that $C_{ik} = \langle \Psi_i, \Psi_k\rangle_{\mathscr{W}_2^{m+1}}$.

Theorem 2.3.2. *For all $u(x) \in \mathscr{W}_2^{m+1}[a, b]$, the series $\sum_{i=1}^{\infty}\langle u(x), \overline{\Psi}_i\rangle\overline{\Psi}_i(x)$ are convergent in the sense of the norm of $\mathscr{W}_2^{m+1}[a, b]$. In contrast if $\{x_i\}_{i=1}^{\infty}$ is dense in $[a, b]$, then the unique solution of the BVP (2.11) and (2.12) is given by*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} B_{ik}\mathcal{F}(x_k, u(x_k), \overline{\Psi}_i(x)). \tag{2.17}$$

Proof : By applying the Theorem (2.3.1), it is easy to see that $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of the space $\mathscr{W}_2^{m+1}[a,b]$. Thus, $u(x)$ can be expanded in the Fourier series about the orthonormal system $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ as $u(x) = \sum_{i=1}^{\infty} \left\langle u(x), \overline{\Psi}_i(x) \right\rangle \Psi_i(x)$. Moreover, the space $\mathscr{W}_2^{m+1}[a,b]$ is Hilbert space, then the series $\sum_{i=1}^{\infty} \left\langle u(x), \overline{\Psi}_i(x) \right\rangle \overline{\Psi}_i(x)$ is convergent in the sense of the norm of $\mathscr{W}_2^{m+1}[a,b]$. Since $\langle v(x), \Phi_i(x) \rangle = v(x_i),\ \forall\ v(x) \in \mathscr{W}_2^{1}[a,b]$, we have

$$\begin{aligned}
u(x) &= \textstyle\sum_{i=1}^{\infty} \langle u(x), \overline{\Psi}_i(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\Psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \langle u(x), \sum_{k=1}^{i} B_{ik} \Psi_k(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\Psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} B_{ik} \langle u(x), \Psi_k(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\Psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} B_{ik} \langle u(x), \mathcal{L}^* \Phi_k(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\Psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} B_{ik} \langle \mathcal{L} u(x), \Phi_k(x) \rangle_{\mathscr{W}_2^{1}} \overline{\Psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} B_{ik} \langle \mathcal{F}(x, u(x)), \Phi_k(x) \rangle_{\mathscr{W}_2^{1}} \overline{\Psi}_i(x), \\
&= \textstyle\sum_{i=0}^{\infty} \sum_{k=1}^{i} B_{ik} \mathcal{F}(x_k, u(x_k)) \overline{\Psi}_i(x).
\end{aligned}$$

□

The $n-$term approximate solution of $u(x)$ denoted by

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} B_{ik} \mathcal{F}(x_k, u(x_k)) \overline{\Psi}_i(x) \tag{2.18}$$

Theorem 2.3.3. *For all $u(x) \in \mathscr{W}_2^{m+1}[a,b]$, $u_n^{(i)}(x)$, are uniformly convergent to $u^{(i)}(x)$, $i = 1, \cdots, m$.*

Proof : By using lemma (2.3.1), for every $x \in [a,b]$, we obtain

$$\begin{aligned}
\left| u_n^{(i)}(x) - u^{(i)}(x) \right| &= \left| \left\langle u_n^{(i)}(x) - u^{(i)}(x), \mathrm{k}_x(x) \right\rangle_{\mathscr{W}_2^{m+1}} \right|, \\
&= \left| \left\langle u_n(x) - u(x), \mathrm{k}_x^{(i)}(x) \right\rangle_{\mathscr{W}_2^{m+1}} \right|, \\
&\leq \left\| \mathrm{k}_x^{(i)}(x) \right\|_{\mathscr{W}_2^{m+1}} \left\| u_n(x) - u(x) \right\|_{\mathscr{W}_2^{m+1}}, \\
&\leq \mathcal{M}_i \left\| u_n(x) - u(x) \right\|_{\mathscr{W}_2^{m+1}} \longrightarrow 0,\ as\ n \longrightarrow \infty.
\end{aligned}$$

So, the approximate solution $u_n(x)$, and $u_n^{(i)}(x)$ converge uniformly to $u(x)$ and its derivative $u^{(i)}(x)$, respectively. □

We will now introduce an effective algorithm to solve BVP (2.13) and (2.14) based on RKHS method, let $\mathrm{k}(x, y)$ be the reproducing kernel function of the space $\mathscr{W}_2^{m+1}[a, b]$.

Algorithm: Use the following steps to approximate the solutions of BVPs (2.13) and (2.14) based on RKHS method.

Input: integer n, real numbers a, b, the functions $\mathrm{k}_1(x, y)$, $\mathrm{k}_2(x, y)$, the differential operator $\mathcal{L}$, the inner product $\langle u(x), v(x)\rangle_{\mathscr{W}_2^{m+1}}$.

Output: Approximate solutions $u_n(x)$ of the BVP (2.13) and (2.14).

- **Stepe A:** Fixed $x \in [a, b]$ and set $y \in [a, b]$;

  For $i = 1, ..., n$ do the following stepes;

  - **stepe 1:** set $x_i = a + \frac{(b-a)i}{n}$;
  - **stepe 2:** if $y \leq x$ then set $\mathrm{k}_x(y) = \mathrm{k}_1(x, y)$ else set $\mathrm{k}_x(y) = \mathrm{k}_2(x, y)$;
  - **stepe 3:** $\Psi_i^{\eta}(x) = \mathscr{L}_y[\mathcal{K}(x, y)]|_{y=x_i}$;

  Output the orthogonal functions system $\Psi_i(x)$.

- **Stepe B:** For $i = 1, ..., n$;

  For $j = 1, ..., i$ set $C_{ij} = \langle \Psi_i, \Psi_j\rangle_{\mathscr{W}_2^{m+1}}$, set $B_{11} = \frac{1}{Sqrt(C_{11})}$;

  Output C_{ij} and B_{11}.

- **Stepe C:** For $i = 2, 3, ..., n$, do stepes;

  - **stepe 1:** For $k = 1, 2, ..., i-1$ set $CC_{ik} = \sum_{m=1}^{k} B_{km} C_{im}$;

- **stepe 2:** For $j = 1, 2, ..., i$, if $j \neq i$;

  then set $B_{ij} = -(\sum_{k=j}^{i-1} CC_{ik}B_{kj}).(C_{ii} - \sum_{k=1}^{i-1}(C_{ik})^2)^{\frac{-1}{2}}$;

  else set $B_{ii} = (C_{ii} - \sum_{k=1}^{i-1}(C_{ik})^2)^{-\frac{1}{2}}$;

  Output the orthogonalization coefficients B_{ij}.

- **Stepe D:** For $i = 1, ..., n$ set $\overline{\Psi}_i(x) = \sum_{k=1}^{i} B_{ik}\Psi_i(x)$;

  Output the orthonormal functions system $\overline{\Psi}_i(x)$;

- **Stepe E:** Set $u_0(x_1) = 0$;

  For $i = 1, ..., n$ do stepes;

  - **stepe 1:** Set $u(x_i) = u_{i-1}(x_i)$;
  - **stepe 2:** Set $\mathcal{B}_i = \sum_{k=1}^{i} B_{ik}\mathcal{F}(x_k, u_{k-1}(x_k))$;
  - **stepe 3:** Set $u_i(x) = \sum_{k=1}^{i} \mathcal{B}_k\overline{\Psi}_k(x)$.

The n-term approximate solution $u_n(x)$ of BVP (2.13) and (2.14) is obtained.

Chapter 3

First and Second Order Fractional Differential Equations

In this chapter, the RKHS method is applied to approximate the solution of a general form of first and second order FDEs. The analytical and approximate solutions are represented in term of series in the RKHS, the $n-$term approximation is obtained and is proved to converge to the analytical solution.

3.1 First Order Fractional Differential Equations

Consider the following first order FDE:

$$\begin{aligned} \mathscr{D}_{*a}^{\alpha}u(x) &= f(x,u(x)),\ a \leq x \leq b,\ 0 < \alpha \leq 1 \\ u(a) &= u_0, \end{aligned} \tag{3.1}$$

such that a, b, and u_0 are real constants, $\mathscr{D}_{*a}^{\alpha}$ denotes the Caputo fractional derivative of order α, $u(x)$ is unknown function to be determined and $f(x, u(x))$ is a linear or nonlinear function depending on the problem discussed. Assume that FDE

(3.1) has a unique solution.

Now, in order to resolve FDE (3.1), we build a several reproducing kernels. The space $\mathscr{W}_2^2[a,b]$ is defined as $\mathscr{W}_2^2[a,b] = \{u : u, u' \textit{ is Abs. C},\ u, u', u'' \in L^2[a,b],\ u(a) = 0\}$. The inner product and the norm in $\mathscr{W}_2^2[a,b]$ are given by

$$\begin{aligned} \langle u, v\rangle_{\mathscr{W}_2^2} &= u(a)v(a) + u'(a)v'(a) + \textstyle\int_a^b u''(t)v''(t)dt. \\ \|u\|_{\mathscr{W}_2^2} &= \sqrt{\langle u, u\rangle}, \end{aligned} \tag{3.2}$$

for all $u, v \in \mathscr{W}_2^2[a,b]$.

Theorem 3.1.1. *The space $\mathscr{W}_2^2[a,b]$ is a reproducing kernel Hilbert space. That is, for every fixed $x \in [a,b]$, there exist $\mathcal{K}_x(y) \in \mathscr{W}_2^2[a,b]$ such that $\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^2} = u(x)$ for all $u(y) \in \mathscr{W}_2^2[a,b]$ and $y \in [a,b]$. The reproducing kernel $\mathrm{k}_x(y)$ can be written as*

$$\mathcal{K}_x(y) = \frac{1}{6}\begin{cases} (y-a)(2a^2 - y^2 + 3x(2+y) - a(6+3x+y)), & y \le x, \\ (x-a)(2a^2 - x^2 + 3y(2+x) - a(6+3y+x)), & y < x. \end{cases} \tag{3.3}$$

Proof : As the same procedure of Example(2.2.1) we have

$$\mathcal{K}_x(y) = \begin{cases} \sum_{i=0}^3 \mathcal{P}_i(x)y^i, & y \le x, \\ \sum_{i=0}^3 \mathcal{Q}_i(x)y^i, & y > x. \end{cases} \tag{3.4}$$

So we will solve the following equations to obtain the unknown coefficients $\mathcal{P}_i(x), \mathcal{Q}_i(x),\ i = 0, 1, 2, 3,$

1) $\mathrm{k}_x(a) = 0$

2) $\mathrm{k}'_x(a) - \mathrm{k}''_x(a) = 0$

3) $\mathrm{k}''_x(b) = 0$

4) $\mathrm{k}'''_x(b) = 0$

5) $\mathrm{k}_x(x+0) = \mathrm{k}_x(x-0)$

6) $\mathrm{k}'_x(x+0) = \mathrm{k}'_x(x-0)$

7) $\mathrm{k}''_x(x+0) = \mathrm{k}''_x(x-0)$

8) $\mathrm{k}'''_x(x+0) - \mathrm{k}'''_x(x-0) = 1$

then substituting $\mathcal{P}_i(x), \mathcal{Q}_i(x)$ in equation (3.4) we have the reproducing kernel function (3.3). □

Let $\mathscr{W}_2^2[a,b] = \{u : \ u, u' \ is \ Abs.C, \ u, u', u'' \in L^2[a,b], \ u(a) = 0\}$, if we are re-defining the inner product (3.3)by

$$\langle u, v\rangle_{\mathscr{W}_2^2} = u(a)v(a) + u(b)v(b) + \int_a^b u''(t)v''(t)dt, \ u, v \in \mathscr{W}_2^2[a,b], \tag{3.5}$$

and the norm $\|u\|_{\mathscr{W}_2^2} = \sqrt{\langle u, u\rangle}, \ u \in \mathscr{W}_2^2[a,b]$, we have the following theorem.

Theorem 3.1.2. *The space $\mathscr{W}_2^2[a,b]$ is a reproducing kernel Hilbert space, That is, for any fixed $x \in [a,b]$, there exists $\mathcal{R}_x(y) \in \mathscr{W}_2^2[a,b]$ such that $\langle u(y), \mathcal{R}_x(y)\rangle_{\mathscr{W}_2^2} = u(x)$ for any $u(y) \in \mathscr{W}_2^2[a,b]$ and $y \in [a,b]$. The reproducing kernel function $\mathcal{R}_x(y)$*

can be written as

$$\mathcal{R}_x(y) = \frac{1}{6(a-b)^2}\begin{cases} \left(\begin{array}{c} -2a^3(b-x)(b-y) + a^2(6+2b^3+x^3+3xy^2-3b(x^2+y^2)) + y \\ (-3b^2x^2+bx^3-b^2y^2+x(6+2b^3+by^2)) - a((-3bx^2+x^3) \\ (b+y) + y(6+2b^3-3b^2y-by^2) + x(6+2b^3+3by^2+y^3)) \end{array}\right), \\ y \le x, \\ \left(\begin{array}{c} -2a^3(b-y)(b-x) + a^2(6+2b^3+y^3+3yx^2-3b(y^2+x^2)) + x \\ (-3b^2y^2+by^3-b^2x^2+y(6+2b^3+bx^2)) - a((-3by^2+y^3) \\ (b+x) + x(6+2b^3-3b^2x-bx^2) + y(6+2b^3+3bx^2+x^3)) \end{array}\right), \\ y > x. \end{cases} \tag{3.6}$$

Proof : As the same procedure of Example (2.2.1) we get

$$\mathcal{R}_x(y) = \begin{cases} \sum_{i=0}^{3} \mathcal{P}_i(x)y^i, & y \le x, \\ \sum_{i=0}^{3} \mathcal{Q}_i(x)y^i, & y > x. \end{cases} \tag{3.7}$$

So we will solve the following equations to obtain the unknown cofficients $\mathcal{P}_i(x), \mathcal{Q}_i(x),\ i = 0, 1, 2, 3$

1) $\mathcal{R}_x(a) = 0$	5) $\mathcal{R}_x(x+0) = \mathcal{R}_x(x-0)$
2) $\mathcal{R}''_x(a) = 0$	6) $\mathcal{R}'_x(x+0) = \mathcal{R}'_x(x-0)$
3) $\mathcal{R}_x(b) - \mathcal{R}'''_x(b) = 0$	7) $\mathcal{R}''_x(x+0) = \mathcal{R}''_x(x-0)$
4) $\mathcal{R}''_x(b) = 0$	8) $\mathcal{R}'''_x(x+0) - \mathcal{R}'''_x(x-0) = 1$

then Substituting $\mathcal{P}_i(x), \mathcal{Q}_i(x)$ in equation(3.7) we have the reproducing kernel function (3.6).

□

The space $\mathscr{W}_2^1[a,b]$ is a complete reproducing kernel space, and we use the reproducing kernel in Exempel(2.2.3)

$$T_x(y) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)].$$

From the definition of the reproducing kernel space $\mathscr{W}_2^2[a,b]$, we get $\mathscr{W}_2^1[a,b] \supset \mathscr{W}_2^2[a,b]$, for all $u \in \mathscr{W}_2^2[a,b]$ and $\|u\|_{\mathscr{W}_2^1} \leq \|u\|_{\mathscr{W}_2^2}$.

To solve equation (3.1), we define a differentian operator $\mathcal{L} : \mathscr{W}_2^2[a,b] \to \mathscr{W}_2^1[a,b]$, such that $\mathcal{L}u(x) = \mathscr{D}_{*a}^{\alpha}$. After homogenization of the initial condition of Equation(3.1), the FDE can be transformed into the corresponding format as follows

$$\begin{cases} \mathcal{L}u(x) &= f(x,u(x)), \\ u(a) &= 0, \end{cases} \tag{3.8}$$

such that $x \in [a,b], u(x) \in \mathscr{W}_2^2[a,b]$ and $f(x,u(x)) \in \mathscr{W}_2^1[a,b]$.

Now, applying the operator $\mathscr{I}_a^{\alpha}$ to both sides we have $u(x) - u(a) = \frac{1}{\Gamma(a)}\int_a^x (x-t)^{\alpha-1} f(x,u(t))dt$, since $u(a) = 0$, then we have

$$u(x) = \mathcal{F}(x,u(x)), \tag{3.9}$$

where $\mathcal{F}(x,u(x)) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(x,u(t))dt$.

So define a differential operator $\mathcal{L} : \mathscr{W}_2^1[a,b] \to \mathscr{W}_2^2[a,b]$ such that $\mathcal{L}u(x) = u(x)$, then it can be converted into the equvalent form as follow

$$\begin{cases} \mathcal{L}u(x) &= \mathcal{F}(x,u(x)), \\ u(a) &= 0. \end{cases} \tag{3.10}$$

where, $x \in [a,b],\ u(x) \in \mathscr{W}_2^2[a,b]$ and $\mathcal{F} \in \mathscr{W}_2^1[a,b]$.

It's clear that $\mathcal{L}u(x) = u(x)$ is bounded operator, now to show that $\mathcal{L}u(x) = \mathcal{D}^{\alpha}_{*a}u(x)$ is bounded operator.

Lemma 3.1.1. *$\mathcal{L}$ is bounded operator from $\mathscr{W}_2^2[a,b]$ to $\mathscr{W}_2^1[a,b]$, where $\mathcal{L}u(x) = \mathcal{D}^{\alpha}_{*a}u(x)$.*

Proof : By reproducing property of $\mathrm{k}_x(y)$ and Schwarz inequality, also since $\mathscr{D}^{\alpha}_{*a}\mathrm{k}_x(y)$ is uniformly bounded about x and y, we can get $|\mathscr{D}^{\alpha}_{*a}u(x)| = |\langle u(y), \mathscr{D}^{\alpha}_{*a}\mathrm{k}_x(y)\rangle_{\mathscr{W}_2^2}| \leq \|\mathscr{D}^{\alpha}_{*a}\mathrm{k}_x(y)\|_{\mathscr{W}_2^2}\|u(y)\|_{\mathscr{W}_2^2} \leq \mathcal{M}_1\|u(y)\|_{\mathscr{W}_2^2}$.
Similarly, we can show that $|\frac{d}{dx}(\mathscr{D}^{\alpha}_{*a}u(x))| \leq \mathcal{M}_2\|u(y)\|_{\mathscr{W}_2^2}$. Hence, $\|\mathcal{L}u(x)\|^2_{\mathscr{W}_2^1} = \|\mathscr{D}^{\alpha}_{*a}u(x)\|^2_{\mathscr{W}_2^1} = \int_a^b(\mathscr{D}^{\alpha}_{*a}u(x))^2 + (\frac{d}{dt}(\mathscr{D}^{\alpha}_{*a}u(t)))^2dt \leq \mathcal{M}\|u(x)\|^2_{\mathscr{W}_2^2}$.
Where $\mathcal{M} = (b-a)(\mathcal{M}_1^2) + (\mathcal{M}_2^2)$. □

Now, we construct an ortogonal function system of $\mathscr{W}_2^2[a,b]$, *let* $\Phi_i(x) = T_{x_i}(x)$ and $\Psi_i(x) = \mathcal{L}^*\Phi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a,b]$, $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ and $T_x(y)$ is the reproducing kernel of $\mathscr{W}_2^1[a,b]$, by the properties of the reproducing kernel $T_x(y)$, we have $\langle u(x), \Phi_i(x)\rangle_{\mathscr{W}_2^1} = u(x_i)$. In term of the properties of $\mathrm{k}_x(y)$, we obtains

$$\langle u(x), \Psi_i(x)\rangle_{\mathcal{W}_2^2} = \langle u(x), \mathcal{L}^*\Phi_i(x)\rangle_{\mathcal{W}_2^2} = \langle \mathcal{L}u(x), \Phi_i(x)\rangle_{\mathcal{W}_2^1} = \mathcal{L}u(x_i). \tag{3.11}$$

Furthemore, $\Psi_i(x)$ can be expressed in the form $\Psi_i(x) = \mathcal{L}_y\mathrm{k}_x(y)|_{y=x_i}$. That is

$$\Psi_i(x) = \mathcal{L}^*\Phi_i(x) = \langle \mathcal{L}^*\Phi_i(x), \mathrm{k}_x(y)\rangle_{\mathcal{W}_2^2} = \langle \Phi_i(x), \mathcal{L}\mathrm{k}_x(y)\rangle_{\mathcal{W}_2^2} = \mathcal{L}_y\mathrm{k}_x(y)|_{y=x_i} = \mathcal{F}(x_i, \mathrm{k}(x_i)), \tag{3.12}$$

For equations (3.8) and (3.10), assume that the inverse operator $\mathcal{L}^{-1}$ exist, therefore, if $\{x_i\}_{i=1}^{\infty}$ is dense in $[a,b]$, then $\{\Psi_i(x)\}_{i=1}^{\infty}$ is the complete function system of the space $\mathcal{W}_2^2[a,b]$. Practice Gram-Schmidt orthonormalization for $\{\Psi_i(x)\}_{i=1}^{\infty}$, we get $\overline{\Psi}_i(x) = \sum_{k=1}^{i} B_{ik}\Psi_k(x)$, where B_{ik} are coefficients of Gram-Schmidt orthonormalization and are given by equation (2.15), and $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ is the orthonormal system in the space $\mathcal{W}_2^2[a,b]$.

Theorem 3.1.3. *Let $\{x_i\}_{i=1}^{\infty}$ be a dense set in $[a,b]$. Then the unique solution of (3.1) on $\mathcal{W}_2^2[a,b]$ is given by*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} B_{ik}\mathcal{F}(x_k, u(x_k))\overline{\Psi}_i(x). \tag{3.13}$$

Proof : According to the orthogonal basis $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ of $\mathcal{W}_2^2[a,b]$, we have

$$\begin{aligned}
u(x) &= \textstyle\sum_{i=1}^{\infty}\langle u(x), \overline{\Psi}_i(x)\rangle_{\mathcal{W}_2^2}\overline{\Psi}_i(x),\\
&= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} B_{ik}\langle u(x), \Psi_k(x)\rangle_{\mathcal{W}_2^2}\overline{\Psi}_i(x),\\
&= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} B_{ik}\langle u(x), \mathcal{L}^*\Phi_k(x)\rangle_{\mathscr{W}_2^2}\overline{\Psi}_i(x),\\
&= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} B_{ik}\langle \mathcal{L}u(x), \Phi_k(x)\rangle_{\mathcal{W}_2^1}\overline{\Psi}_i(x),\\
&= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} B_{ik}\langle \mathcal{F}(x, u(x)), \Phi_k(x)\rangle_{\mathcal{W}_2^1}\overline{\Psi}_i(x),\\
&= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} B_{ik}\mathcal{F}(x_k, u(x_k))\overline{\Psi}_i(x).
\end{aligned}$$

□

We denote the $n-$term approximate solution to $u(x)$ by

$$u(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} B_{ik}\mathcal{F}(x_k, u(x_k))\overline{\Psi}_i(x). \tag{3.14}$$

By using Theorem (2.1.3) we having the following lemma

Lemma 3.1.2. *The approximate solution $u_n(x)$ and its derivative $u_n'(x)$ are both uniformly convergent.*

3.2 Second Order Fractional Differential Equations

The RKHS method is used in this section to get a numerical solution for initial and boundary second-order FDEs. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in terms of series in the space $\mathcal{W}_2^3[a,b]$.

3.2.1 IVP Second Order FDEs

Consider the IVP second order FDE of the following form:

$$\mathscr{D}_{*a}^{\alpha}u(x) = \mathcal{F}(x, u(x), u'(x)), \ a \leq x \leq b, \ 1 < \alpha \leq 2, \tag{3.15}$$

$$u(a) = u_0, \ u'(a) = u_1.$$

where a, b, u_0 and u_1 are real constants, $\mathscr{D}_{*a}^{\alpha}$ denotes the Caputo fractional derivative of order α, $f(x, u(x), u'(x))$ is a linear or nonlinear function depending on the problem discussed, and $u(x)$ is unknown function to be determined.

We construct a reproducing kernel space $\mathcal{W}_2^3[a,b]$ in which every function satisfies $u(a) = 0, \ u'(a) = 0$. The reproducing kernel Hilbert space $\mathcal{W}_2^3[a,b]$ is defined as $\mathscr{W}_2^3[a,b] = \{u : u, u', u'' \textit{ are Abs. C}, \ u, u', u'', u''' \in L^2[a,b], \ u(a) = u'(a) = 0\}$. The inner product and the norm are given respectively by

$$\langle u, v\rangle_{\mathscr{W}_2^3} = u(a)v(a) + u'(a)v'(a) + u''(a)v''(a) + \int_a^b u'''(t)v'''(t)dt. \tag{3.16}$$

and $\|u\|_{\mathscr{W}_2^3} = \sqrt{\langle u, u\rangle_{\mathscr{W}_2^3}}$, where $u, v \in \mathscr{W}_2^3[a,b]$.

Theorem 3.2.1. *The space $\mathscr{W}_2^3[a,b]$ is a reproducing Kernel space. That is, for every fixed $x \in [a,b]$ and all $u(x) \in \mathscr{W}_2^3[a,b]$, there exist $\mathrm{k}_x(y) \in \mathscr{W}_2^3[a,b]$, $y \in [a,b]$*

such that $\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^3} = u(x)$, *and the reproducing kernel* $\mathrm{k}_x(y)$ *can be presented by*

$$\mathrm{k}_x(y) = \frac{1}{120}\begin{cases} (a-y)^2\begin{pmatrix} -6a^3 - 5xy^2 + y^3 + 10x^2(3+y) + 3a^2(10 \\ +5x+y) - 2a(5x^2 - y^2 + 5x(6+y) \end{pmatrix}, & y \le x, \\ (a-x)^2\begin{pmatrix} -6a^3 - 5yx^2 + x^3 + 10y^2(3+x) + 3a^2(10 \\ +5y+x) - 2a(5y^2 - x^2 + 5y(6+x) \end{pmatrix}, & y > x. \end{cases} \tag{3.17}$$

Proof : As the same procedure of Example (2.2.1) we have

$$\mathrm{k}_x(y) = \begin{cases} \sum_{i=0}^{5} \mathcal{P}_i(x) y^i, \ y \le x, \\ \sum_{i=0}^{5} \mathcal{Q}_i(x) y^i, \ y > x. \end{cases} \tag{3.18}$$

and we have the following equations

1) $\mathrm{k}_x(a) = 0$

2) $\mathrm{k}'_x(a) = 0$

3) $\mathrm{k}''_x(a) - \mathrm{k}'''_x(a) = 0$

4) $\mathrm{k}'''_x(b) = 0$

5) $\mathrm{k}^{(4)}_x(b) = 0$

6) $\mathrm{k}^{(5)}_x(b) = 0$

7) $\mathrm{k}_x(x+0) = \mathrm{k}_x(x-0)$

8) $\mathrm{k}'_x(x+0) = \mathrm{k}'_x(x-0)$

9) $\mathrm{k}''_x(x+0) = \mathrm{k}''_x(x-0)$

10) $\mathrm{k}'''_x(x+0) = \mathrm{k}'''_x(x-0)$

11) $\mathrm{k}^{(4)}_x(x+0) = \mathrm{k}^{(4)}_x(x-0)$

12) $\mathrm{k}^{(5)}_x(x+0) - \mathrm{k}^{(5)}_x(x-0) = 1$

We find the unknown coefficients $\mathcal{P}_i, \ \mathcal{Q}_i, \ i = 0, 1, ...5$ by using Mathematica 11.0, then substituting these coefficients in (3.18), hence (3.17) is obtained. □

Let $\mathscr{W}_2^3[a,b] = \{u : u, u', u''$ *are Abs. C*, $u, u', u'', u''' \in L^2[a,b], \ u(a) = u'(a) = 0\}$, but if we are re-defining the inner product (3.16) by

$$\langle u, v\rangle_{\mathscr{W}_2^3} = u(a)v(a) + u'(a)v'(a) + u(b)v(b) + \int_a^b u'''(t)v'''(t)dt. \tag{3.19}$$

and the norm $\|u\|_{\mathscr{W}_2^3} = \sqrt{\langle u, u\rangle_{\mathscr{W}_2^3}}$, $u, v \in \mathscr{W}_2^3[a, b]$, we get the following theorem.

Theorem 3.2.2. *The space $\mathscr{W}_2^3[a, b]$ is a reproducing kernel space. That is, for every fixed $x \in [a, b]$ and all $u(x) \in \mathscr{W}_2^3[a, b]$, there exist $\mathcal{R}_x(y) \in \mathscr{W}_2^3[a, b]$, $y \in [a, b]$ such that $\langle u(y), \mathcal{R}_x(y)\rangle$, and the reproducing kernel $\mathcal{R}_x(y)$ can be presented by*

$$\mathcal{R}_x(y) = \begin{cases} g(x, y), \;\; y \le x, \\ g(y, x), \;\; y > x. \end{cases} \tag{3.20}$$

where $g(x, y) = -\dfrac{1}{120(a-b)^4}(a-y)^2(10b^4x^3 - 5b^3x^4 + b^2x^5 + 4a^5$
$(b-x)(b-y) - a^4(b-x)(b-y)(14b+7x-y) + 5b^4xy^2 - b^4y^3 + x^2(-120 - 6b^5 - 5b^3y^2$
$+b^2y^3) + 2a^3(b-y)(8b^3 + b^2(5x-y) + xy(-x+y) - b(13x^2 - 2xy + y^2)) + a^2(-120 - 6b^5$
$+ x^5 + x^2y^3 + b^4(-25x + 7y) + b^3(25x^2 + 20xy + 3y^2) + b^2(10x^3 - 27x^2y + 6xy^2$
$- 5y^3) + bx(-5x^3 - 9xy^2 + 4y^3)) - 2a(10b^3x^3 - 5b^2x^4 + bx^5 + b^3(b-2y)y^2$
$+ bx^2y(-5b^2 - 6by + y^2 + x(-120 - 6b^5 + 5b^4y + 5b^3y^2 + b^2y^3)))$.

Proof : As the same procedure of example(2.2.1)

$$\mathcal{R}_x(y) = \begin{cases} \sum_{i=0}^{5} \mathcal{P}_i(x)y^i, \;\; y \le x, \\ \sum_{i=0}^{5} \mathcal{Q}_i(x)y^i, \;\; y > x. \end{cases} \tag{3.21}$$

and we have the following equations

1) $\mathcal{R}_x(a) = 0$

2) $\mathcal{R}'_x(a) = 0$

3) $\mathcal{R}'''_x(a) = 0$

4) $\mathcal{R}_x(b) + \mathcal{R}^{(5)}_x(b) = 0$

5) $\mathcal{R}'''_x(b) = 0$

6) $\mathcal{R}^{(4)}_x(b) = 0$

7) $\mathcal{R}_x(x+0) = \mathcal{R}_x(x-0)$

8) $\mathcal{R}'_x(x+0) = \mathcal{R}'_x(x-0)$

9) $\mathcal{R}''_x(x+0) = \mathcal{R}''_x(x-0)$

10) $\mathcal{R}'''_x(x+0) = \mathcal{R}'''_x(x-0)$

11) $\mathcal{R}^{(4)}_x(x+0) = \mathcal{R}^{(4)}_x(x-0)$

12) $\mathcal{R}^{(5)}_x(x+0) - \mathcal{R}^{(5)}_x(x-0) = 1$

We find the unknown coefficients $\mathcal{P}_i,\ \mathcal{Q}_i,\ i = 0, 1, \cdots, 5$, by using Mathematica 11.0, then substituting coefficients in(3.21), hence (3.20) is obtained. □

To solve equation (3.15) we define a differentail operator $\mathcal{L} : \mathscr{W}_2^3[a,b] \to \mathscr{W}_2^1[a,b]$, $\mathcal{L}u(x) = \mathscr{D}^{\alpha}_{*a}$. After homogenizing the initial condition of equation (3.15), then it can be transformed into the equivalent form:

$$\begin{cases} \mathcal{L}u(x) = f(x, u(x), u'(x)), \\ u(a) = u'(a) = 0. \end{cases} \tag{3.22}$$

where $x \in [a,b],\ u(x) \in \mathscr{W}_2^3[a,b]$ and $f(x, u(x), u'(x)) \in \mathscr{W}_2^1[a,b]$.

Now, applying the operator $\mathscr{I}^{\alpha}_a$ of both sides we have

$$u(x) - \sum_{k=0}^{1} \frac{u^k(a)}{k!}(x-a)^k = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(x, u(t), u'(t))dt, \tag{3.23}$$

since $u^{(i)}(a) = 0,\ i = 0, 1$, then we get

$$u(x) = \mathcal{F}(x, u(x), u'(x)), \tag{3.24}$$

where $\mathcal{F}(x, u(x), u'(x)) = \dfrac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(x, u(t), u'(t))dt$.

So define a differential operator $\mathcal{L} : \mathscr{W}_2^3[a,b] \to \mathscr{W}_2^1[a,b],\ \mathcal{L}u(x) = u(x)$, the FDE

(3.15) can be converted into the equivalent form as follow:

$$\begin{cases} \mathcal{L}u(x) = f(x, u(x), u'(x)), \\ u(a) = u'(a) = 0, \end{cases} \tag{3.25}$$

where $x \in [a, b],\ u(x) \in \mathscr{W}_2^3[a, b]$, and $\mathcal{F}(x, u(x), u'(x)) \in \mathscr{W}_2^1[a, b]$.

3.2.2 BVP Second Order FDEs

Consider the BVP second order FDE of the following form:

$$\mathscr{D}_{*a}^{\alpha} u(x) = f(x, u(x), u'(x)),\ a \leq x \leq b,\ 1 < \alpha \leq 2, \tag{3.26}$$

$$u(a) = u_0,\ u(b) = u_1 \tag{3.27}$$

where a, b, u_0 and u_1 are real constants, $\mathscr{D}_{*a}^{\alpha}$ denotes the Caputo fractional derivative of order α, $f(x, u(x), u'(x))$ is a linear or nonlinear function depending on the problem discussed, and $u(x)$ is unknown function to be determined.

We construct a reproducing kernel space

$$\mathscr{W}_2^3[a, b] = \{u:\ u, u', u''\ \textit{are Abs. C},\ u, u', u'', u''' \in L^2[a, b],\ u(a) = u(b) = 0\},$$

in which every function satisfies $u(a) = 0,\ u(b) = 0$. The inner product and the norm are given, respectively by

$$\langle u, v \rangle_{\mathscr{W}_2^3} = u(a)v(a) + u'(a)v'(a) + u(b)v(b) + \int_a^b u'''(t)v'''(t)dt, \tag{3.28}$$

and the norm $\|u\|_{\mathscr{W}_2^3} = \sqrt{\langle u, u \rangle_{\mathscr{W}_2^3}}$ where $u, v \in \mathscr{W}_2^3[a, b]$.

Theorem 3.2.3. *The space* $\mathscr{W}_2^3[a, b]$ *is a reproducing kernel space. That is, for* *every fixed* $x \in [a,b]$ *and all* $u(x) \in \mathscr{W}_2^3[a,b]$, *there exist* $\mathcal{S}_x(y) \in \mathscr{W}_2^3[a,b]$, $y \in [a,b]$ *such that* $\langle u(y), \mathcal{S}_x(y) \rangle_{\mathscr{W}_2^3} = u(x)$, *and the reproducing kernel* $\mathcal{S}_x(y)$ *can be presented by*

$$\mathcal{S}_x(y) = \begin{cases} h(x,y), \ y \leq x, \\ h(y,x), \ y > x. \end{cases} \tag{3.29}$$

where $h(x,y) = \dfrac{1}{120(a-b)^2}(a-y)(-4a^4(b-x)(b-y) - 6b^3x^2y + a^3(b-x)(b-y)(6b+7x+3y) + x^2y(-120+x^3+y^3) - 3a^2(b-y)(xy(-3x+y) + 2b^2(2x+y) - b(4x^2-xy+y^2)) - 5bx(-24y+x^3y+x(-24+y^3)) + b^2(10x^3y - y^4 + 5x(-24+y^3)) + a(6b^3x(x+2y) - b^2(-120+10x^3+12x^2y+15xy^2+y^3) + x(-x^4+xy^3-2y(-60+y^3)) + b(-120x+5x^4+15x^2y_2+2y(-60+y^3))))$.

Proof : As the same procedure of Example(2.2.1), we have

$$\mathcal{S}_x(y) = \begin{cases} \sum_{i=0}^{5} \mathcal{P}_i(x)y^i, \ y \leq x, \\ \sum_{i=0}^{5} \mathcal{Q}_i(x)y^i, \ y > x. \end{cases} \tag{3.30}$$

and we have the following equations

1) $\mathcal{S}_x(a) = 0$	7) $\mathcal{S}_x(x+0) = \mathcal{S}_x(x-0)$
2) $\mathcal{S}_x(b) = 0$	8) $\mathcal{S}'_x(x+0) = \mathcal{S}'_x(x-0)$
3) $\mathcal{S}'_x(a) + \mathcal{S}_x^{(4)}(a) = 0$	9) $\mathcal{S}''_x(x+0) = \mathcal{S}''_x(x-0)$
4) $\mathcal{S}'''_x(a) = 0$	10) $\mathcal{S}'''_x(x+0) = \mathcal{S}'''_x(x-0)$
5) $\mathcal{S}_x^{(3)}(b) = 0$	11) $\mathcal{S}_x^{(4)}(x+0) = \mathcal{S}_x^{(4)}(x-0)$
6) $\mathcal{S}_x^{4}(b) = 0$	12) $\mathcal{S}_x^{(5)}(x+0) - \mathcal{S}_x^{(5)}(x-0) = 1$

We find the unknown coefficients $\mathcal{P}_i$, $\mathcal{Q}_i$, $i = 0, 1, \cdots, 5$, by using Mathematica 11.0, then substituting coefficients in (3.30), hence (3.29) is obtained. □

The space $\mathscr{W}_2^1[a,b]$ is also a complete reproducing kernel space and we use the reproducing kernel in Exempel(2.2.3)

$$T_x(y) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)].$$

From the definition of the reproducing kernel space $\mathscr{W}_2^1[a,b]$ and $\mathscr{W}_2^3[a,b]$, we get $\mathscr{W}_2^1[a,b] \supset \mathscr{W}_2^3[a,b]$, for any $u \in \mathscr{W}_2^3[a,b]$, and $\|u\|_{\mathscr{W}_2^1[a,b]} \leq \|u\|_{\mathscr{W}_2^3[a,b]}$.
To solve equation (3.26) we define a differential operator $\mathcal{L} : \mathscr{W}_2^3[a,b] \to \mathscr{W}_2^1[a,b]$, such that $\mathcal{L}u(x) = \mathscr{D}_{*a}^{\alpha}u(x)$. After homogenization of the bondary conditions of equation (3.26), it can be converted into the equivalent from as follow:

$$\mathcal{L}u(x) = f(x, u(x), u'(x)), \;\; a \leq x \leq b, \tag{3.31}$$

$$u(a) = u(b) = 0. \tag{3.32}$$

where $u(x) \in \mathscr{W}_2^3[a,b]$ and $f(x,u,u') \in \mathscr{W}_2^1$.

3.3 Numerical Examples

In this section, five numerical examples are given to demonstrate the accuracy of this method. The computations are performed by Mathematica 11.0. We compare the results by this method with the exact solution of each example.

Example 3.3.1. *Consider the following linear FDE:*

$$\begin{cases} \mathscr{D}_{*0}^{0.5} = \sqrt{\pi}\; x, \quad 0 \leq x \leq 1, \\ u(0) = 0, \end{cases}$$

with the exact solution $u(x) = x^{1.5}$.

By using the RKHS method, taking $n = 30$, $x_i = \frac{i}{n}$, $i = 1, \cdots, n$, *with the reproducing kernel* $\mathrm{k}_x(y)$ *on* $[0, 1]$, *the numerical results are given in Table (3.1) and Figure (3.1).*

Table 3.1: Numerical results for $u(x)$ of Example 3.3.1.

x	Exact Solution	Approximate solution	Absolute Error
0.	0	0	0
0.1	0.03162278	0.03162278	$1.90125693 \times 10^{-15}$
0.2	0.08944272	0.16431677	$3.80251386 \times 10^{-15}$
0.3	0.16431677	0.16431677	$3.80251386 \times 10^{-15}$
0.4	0.25298221	0.25298221	$1.54321000 \times 10^{-14}$
0.5	0.35355339	0.35355339	$6.23945340 \times 10^{-14}$
0.6	0.46475800	0.46475800	$1.05249143 \times 10^{-13}$
0.7	0.58566202	0.58566202	$2.11164419 \times 10^{-13}$
0.8	0.71554175	0.71554175	$2.36366482 \times 10^{-13}$
0.9	0.85381497	0.85381497	$3.41948692 \times 10^{-13}$
1.	1	1	$2.36477504 \times 10^{-13}$

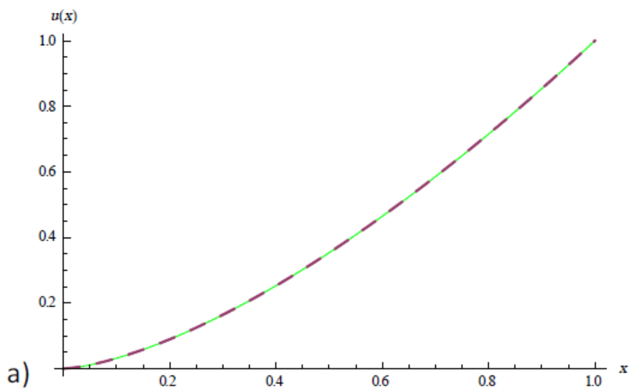

Figure 3.1: Exact and approximate solution $u(x)$

Example 3.3.2. *Consider the fractional Riccati equation:*

$$\begin{cases} \mathscr{D}^{\alpha}_{*0} u(x) = 1 + x^2 - u^2(x), \quad 0 \le x \le 1, \ 0 \le \alpha \le 1, \\ u(0) = 1, \end{cases}$$

The exact solution, when $\alpha = 1$, *is* $u(x) = x + \frac{e^{-x^2}}{1+\int_0^x e^{-t^2} dt}$. *Using the RKHS method, taking* $n = 25$, $x_i = \frac{i}{n}$, $i = 1, \cdots, n$ *with the reproducing kernel* $\mathrm{R}_x(y)$ *on* $[0, 1]$, *the* *numerical results are given in Table (3.2) and Figure (3.2).*

Table 3.2: Numerical results for $u(x)$ of Example 3.3.2.

x	Exact Solution	Approximate solution	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$	Absolute Error $\alpha = 1$
0.	1	1	1	1	1	0
0.1	1.00031731	1.00037230	1.00145635	1.00251261	1.00358873	$5.49913502 \times 10^{-5}$
0.2	1.00241983	1.00251362	1.00501586	1.00742158	1.00982945	$9.37945689 \times 10^{-5}$
0.3	1.00779459	1.00793879	1.01206811	1.01595610	1.01977952	$1.44202866 \times 10^{-4}$
0.4	1.01765088	1.01785241	1.02372103	1.02911568	1.03433001	$2.01534032 \times 10^{-4}$
0.5	1.03295758	1.03321888	1.04064780	1.04740649	1.05389405	$2.61308818 \times 10^{-4}$
0.6	1.0544669	1.05478664	1.06364147	1.07171437	1.07944657	$3.19833581 \times 10^{-4}$
0.7	1.08272748	1.08310095	1.09302288	1.10223924	1.11112432	$3.73469815 \times 10^{-4}$
0.8	1.11809254	1.11851185	1.12932240	1.13988924	1.15023992	$4.19300556 \times 10^{-4}$
0.9	1.16072397	1.16117856	1.17264233	1.18458423	1.19651741	$4.54589936 \times 10^{-4}$
1	1.21059901	1.21107650	1.20522568	1.20188362	1.20053356	$4.77488710 \times 10^{-4}$

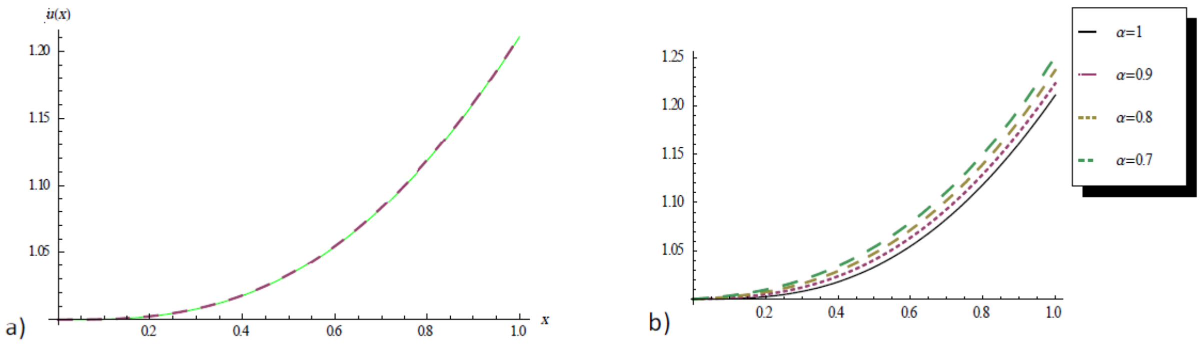

Figure 3.2: (a) Exact and approximate solution $u(x)$, (b) Approximate solution of $u(x)$ for different values of α.

Example 3.3.3. *Consider the following linear FDE:*

$$\begin{cases} \mathscr{D}_{*1}^{\alpha} u(x) = 1 - u(x), \quad 1 \leq x \leq 3, \ 1 \leq \alpha \leq 2, \\ u(1) = u'(1) = 0. \end{cases}$$

The exact solution, when $\alpha = 2$ is $u(x) = 1 - \cos(1 - x)$. Using the RKHS method, taking $n = 100$, $x_i = 1 + \frac{2i}{n}$, with the reproducing kernel $\mathrm{k}_x(y)$ on $[1, 3]$, the numerical results are given in Table (3.3) and Figure (3.3).

Table 3.3: Numerical results for $u(x)$ of Example 3.3.3.

x	Exact Solution	Approximate solution	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	Absolute Error $\alpha = 1$
1.	0	0	0	0	0	0
1.2	0.01993342	0.01989579	0.02545006	0.03349014	0.04526083	$3.76281416 \times 10^{-5}$
1.4	0.07893900	0.07886271	0.09535639	0.12028719	0.15793205	$7.62923692 \times 10^{-5}$
1.6	0.17466439	0.17455123	0.20343728	0.24988894	0.32266385	$1.13155137 \times 10^{-4}$
1.8	0.30329329	0.30314666	0.34355708	0.41387356	0.52846831	$1.46629448 \times 10^{-4}$
2	0.45969769	0.45952248	0.50904424	0.60406706	0.76537680	$1.75218634 \times 10^{-4}$
2.2	0.63764225	0.63744466	0.6928211	0.81249286	1.02414375	$1.97582491 \times 10^{-4}$
2.4	0.83003286	0.82982026	0.88767058	1.03156375	1.29632951	$2.12598697 \times 10^{-4}$
2.6	1.02919952	1.02898011	1.08653013	1.25430708	1.57444112	$2.19416757 \times 10^{-4}$
2.8	1.22720209	1.22698459	1.28277018	1.47455971	1.85204723	$2.17501945 \times 10^{-4}$
3	1.41614684	1.41594017	1.50530711	1.75535906	2.23539176	$2.06667109 \times 10^{-4}$

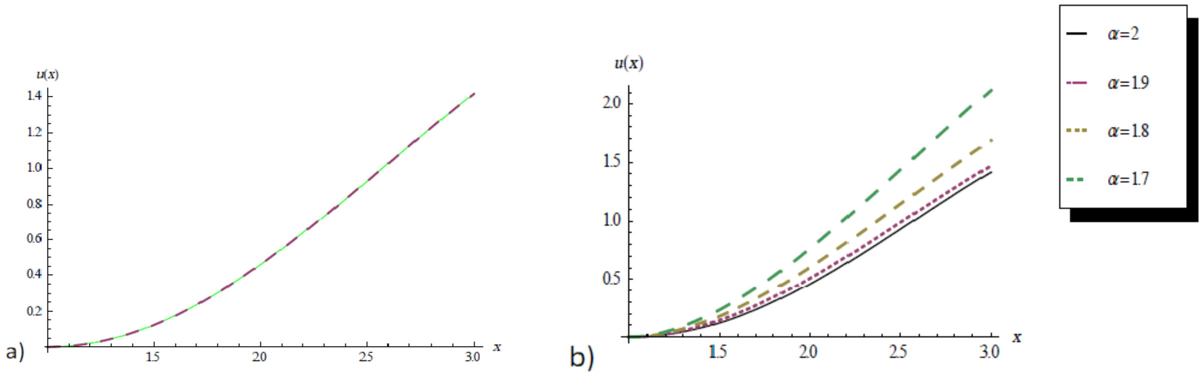

Figure 3.3: (a) Exact and approximate solution $u(x)$, (b) Approximate solution of $u(x)$ for different values of α.

Example 3.3.4. *Consider the following nonlinear FDE:*

$$\begin{cases} \mathscr{D}_{*0}^{1.4} u(x) = x u^2(x) + \frac{25}{4}\Gamma(0.6) x^{1.6} - x^7, \quad 0 \le x \le 1, \\ u(0) = u'(0) = 0, \end{cases}$$

The exact solution is $u(x) = x^3$. *Using the RKHS method, taking* $n = 15$, $x_i = \frac{i}{n}$, *with the reproducing kernel* $\mathrm{k}_x(y)$ *on* $[0, 1]$, *the numerical results are given in Table (3.4) and Figure (3.4).*

Table 3.4: Numerical results for $u(x)$ of Example 3.3.4.

x	Exact Solution	Approximate solution	Absolute Error
0.	0	0	0
0.1	0.00100000	0.00099504	$4.96443711 \times 10^{-6}$
0.2	0.00200000	0.00799990	$9.50830642 \times 10^{-9}$
0.3	0.27000000	0.02700035	$3.52173496 \times 10^{-7}$
0.4	0.06400000	0.06399977	$2.25374170 \times 10^{-7}$
0.5	0.12500000	0.12499930	$6.99193108 \times 10^{-7}$
0.6	0.21600000	0.21599879	$1.21083166 \times 10^{-6}$
0.7	0.34300000	0.34299894	$1.05900564 \times 10^{-6}$
0.8	0.51200000	0.51199597	$4.02729866 \times 10^{-6}$
0.9	0.72900000	0.72897741	$2.25866151 \times 10^{-5}$
1	1.00000000	0.99998936	$1.06446764 \times 10^{-5}$

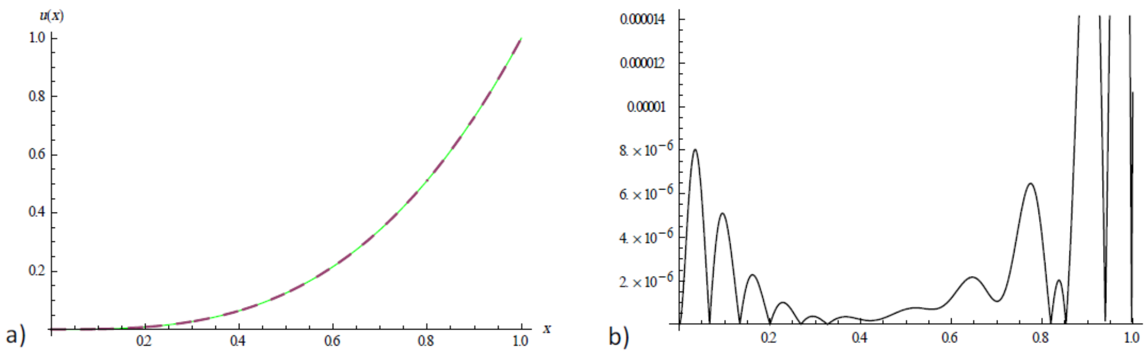

Figure 3.4: (a) Exact and approximate solution $u(x)$, (b) Absolute error for Example 3.3.4

Example 3.3.5. *Consider the following linear boundary FDE:*

$$\begin{cases} \mathscr{D}^{\alpha}_{*0}u(x) = 1 - u(x), \quad 0 \leq x \leq 1,\ 1 \leq \alpha \leq 2, \\ u(0) = u(1) = 0. \end{cases}$$

The exact solution, when $\alpha = 2$ is $u(x) = 1 - \cos(x) - \sin(x)\tan(\frac{1}{2})$. Using the RKHS method, taking $n = 50$, $x_i = \frac{i}{n}$, with the reproducing kernel $\mathrm{S}_x(y)$ on $[0,1]$, the numerical results are given in Table (3.5) and Figure (3.5).

Table 3.5: Numerical results for $u(x)$ of Example 3.3.5.

x	Exact Solution	Approximate solution	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	Absolute Error $\alpha = 1$
1.	0	0	0	0	0	0
0.1	-0.04954341	-0.04945350	-0.05382040	-0.06032754	-0.06965920	$8.99036116 \times 10^{-5}$
0.2	-0.08860012	-0.08846678	-0.09485964	-0.10496952	-0.11990704	$1.33352215 \times 10^{-4}$
0.3	-0.116779914	-0.11664899	-0.12357628	-0.13540199	-0.15348222	$1.30919176 \times 10^{-4}$
0.4	-0.13380120	-0.13370082	-0.14018375	-0.15237329	-0.17171690	$1.00381918 \times 10^{-4}$
0.5	-0.13949393	-0.13943117	-0.14491005	-0.15651318	-0.17565023	$6.27545731 \times 10^{-5}$
0.6	-0.13380120	-0.13376886	-0.13805021	-0.14844555	-0.16622644	$3.23473623 \times 10^{-5}$
0.7	-0.11677991	-0.11676583	-0.11997783	-0.12881853	-0.14435333	$1.40826410 \times 10^{-5}$
0.8	-0.08860012	-0.08859438	-0.09114448	-0.09831196	-0.11092174	$5.74811049 \times 10^{-6}$
0.9	-0.0495434	-0.04954106	-0.05207626	-0.05763784	-0.06680999	$2.34876695 \times 10^{-6}$
1	0	0	0	0	0	0

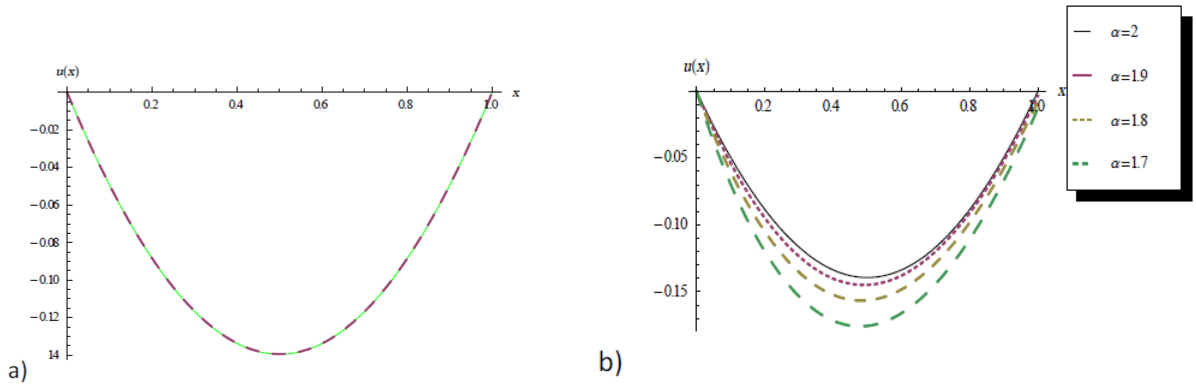

Figure 3.5: (a) Exact and approximate solution $u(x)$, (b) Approximate solution of $u(x)$ for different values of α

Conclusion and Future Recommendations

In this thesis, we apply the R.K.H.S.M to obtain the approximate solution of the fractional differential equations of first and second order.
The numerical results are given to compare this approach with the exact solution for different values of the order derivative α in the Caputo sense.
The results of examples which are shown in tables end figures, proved the effeciency and the accuracy of the method. The analytical and the approximate solutions are presented in the form of series in the space $\mathscr{W}_2^m[a, b]$. Additionally, the approximate solution and its derivatives are uniformly convergent to the exact solution and its derivatives respectiveley. By given a new form of the reproducing kernel function and different R.K.H.S to deal with the initial and boundary condition, the our method can be used to deal with other types of F.D.E's with non- classical conditions.
In the future, we try to use the R.K.H.S.M for:

- solving the mixed differential equations with integer and fractional orders.
- solving a system of F.D.E's of first and second order.
- solving the F.D.E's with higher order.

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Abstract

In this thesis, based on the reproducing kernel Hilbert space method (RKHSM) an efficient algorithm is presented for solving ordinary differential equations of fractional order. We applied RKHSM to obtain approximate solution for a general form of first and second order fractional differential equations. The analytical and approximate solutions are represented in the form of series in the reproducing kernel space $\mathscr{W}_2^m[a,b]$. The $n-$term approximation and all its derivatives are obtained and proved to converge uniformly to the analytical solution and all its derivatives, respectively. The proposed method has an advantage that it is possible to pick any point in the interval of integration. Numerical examples are given to demonstrate the computation efficiency of the presented method. The results of applying this method to the studied cases show the high accuracy, simplicity and efficiency of the approach.

ملخص

في هذه الرسالة، نقوم بدراسة طريقة استنساخ نواة فضاء هلبرت وتطبيقها على المعادلات التفاضلية العادية ذات الرتبة الكسرية من الدرجة الأولى، والثانية للحصول على حلول تقريبية لهذه المعادلات.

يمثل الحل التحليلي والتقريبي على شكل سلسلة من فضاء سو بوليف. كما نوضح أن الحل التقريبي يتقارب بشكل منتظم من الحل الصحيح. كما قمنا بدراسة مجموعة من الأمثلة التي عرضت الفعالية والكفاءة لطريقة استنساخ النواة.

Le Résumé

Cette thése présente une approche basée sur la méthode de (RKHSM) pour résoudre efficacement des équations différentielles ordinaires d'ordre fractionnaire. Cette approche consistait á utiliser la méthode RKHSM afin d'obtenir une solution approximative aux équations différentielles fractionnaires du premier et du second ordre, dans une forme générale. Les solution analytiques et approximatives sont représentées sous forme de séries dans l'espace de noyau reproducteur $\mathscr{W}_2^m[a, b]$. L'approximation en n termes et ses dérivées sont obtenues et prouvées converger uniformément vers la solution analytique et toutes ses dérivées respectivement. L'avantage de la méthode proposée est la flexibilité de choisir n'importe quel point dans l'intervalle d'intégration. Des exemples numériques sont fournis pour démontrer l'efficacité de calcul de la méthode présentée. Les résultats obtenus montrent une grande précision, simplicité et efficacité de cette approche dans les cas étudiés.